Self-generated flux in Josephson junctions with alternating critical current density

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We study an interesting state with a self-generated magnetic flux in a long Josephson junction. The critical current density \( j_c \) is assumed to alternate randomly along the tunnel contact on a length scale much less than the local Josephson penetration depth. The phase difference is then a sum of an alternating term and a smooth term which self-consistently contribute to the formation of a stationary state with self-generated flux. Two types of Josephson vortices are found in this state, one with magnetic flux \( \Phi_1 < \Phi_0 / 2 \) and another one with magnetic flux \( \Phi_2 = \Phi_0 - \Phi_1 > \Phi_0 / 2 \).

Recently Mannhart et al.\(^1\) observed a self-generated magnetic flux in asymmetric 45° [001] tilt grain boundaries in YBa\(_2\)Cu\(_3\)O\(_{7-x}\) films in zero applied field. The authors explained qualitatively the existence of this flux using a model\(^2\) based on the assumption of a \( d_{z^2-r^2} \) symmetry component of the order parameter and on the observation of facets with alternating orientation forming the grain boundary.\(^3\) This model introduces a tunneling current density \( j(x) \propto \sin(\varphi(x) - \alpha(x)) \), where \( x \) is along the grain-boundary line, \( \varphi(x) \) is the phase difference caused by the flux, and \( \alpha(x) \) is the phase difference caused by the misalignment of anisotropic \( d_{z^2-r^2} \)-wave superconductors. The value of \( \alpha(x) \) is determined by the orientation of the facets, for asymmetric 45° grain boundaries \( \alpha(x) = 0 \) or \( \pi \), therefore \( j(x) \propto \cos(\alpha(x)) \sin \varphi(x) \).

The self-generated flux \( \Phi_s(x) \) observed by Mannhart et al.\(^1\) is randomly distributed along a meandering grain-boundary line. The flux \( \Phi_s(x) \) changes its sign randomly, the amplitude of \( \Phi_s(x) \) variations is much less than the flux quantum \( \Phi_0 \), and the average \( \langle \Phi_s \rangle \) is nearly zero.

In the framework of a model relating \( j(x) \) to orientation of facets,\(^2\) a meandering grain boundary results in an alternating critical current density \( j_c(x) \propto \cos \alpha \). The length scales for variations in \( \Phi_s(x) \) and \( j_c(x) \) are of the same order. This leads to a nonzero average \( \langle j \rangle \) of the tunneling current density \( j(x) \) induced by an alternating flux \( \Phi_s(x) \). Namely, the current density \( j(x) \) is equal to \( j_c(x) \) for \( \xi(x) = 2\pi \Phi_s(x) / \Phi_0 \) is the alternating phase difference caused by the flux \( \Phi_s(x) \). Since \( j_c(x) \) and \( \sin \xi(x) \) alternate on the same typical scale, their product has a nonzero average \( \langle j \rangle \). This consequence, which has not been mentioned in the qualitative consideration,\(^1\) is important for understanding the self-generated flux \( \Phi_s(x) \).

In a stationary state, however, \( \langle j \rangle \) has to be zero. The contradiction is resolved if a smooth phase shift \( \psi(x) \) arises simultaneously with the flux \( \Phi_s(x) \). The additional phase \( \psi(x) \) causes an additional smooth current density \( j_s(x) \), which compensates the nonzero current induced by the fast alternating phase \( \xi(x) \). The average current density \( \langle j \rangle \) is then zero and a self-generated flux \( \Phi_s(x) \) is established self-consistently. Namely, on the background of an alternating critical current density \( j_c(x) \) the alternating flux \( \Phi_s(x) \) causes a smooth phase shift \( \psi(x) \), and the smooth phase shift \( \psi(x) \) in its turn determines the alternating flux \( \Phi_s(x) \).

In this paper we study this unusual stationary state with a self-generated flux \( \Phi_s(x) \) in a Josephson junction with an alternating critical current density. We derive equations determining the alternating phase difference \( \xi(x) \) and the smooth phase shift \( \psi(x) \). From these equations we obtain a spontaneous flux \( \Phi_s(x) \) occurring when the average energy density of the self-generated magnetic field exceeds that of a critical value proportional to the average critical current density. We find two types of Josephson vortices in a state with a self-generated flux: one with flux \( \Phi_1 < \Phi_0 / 2 \) and one with flux \( \Phi_2 = \Phi_0 - \Phi_1 > \Phi_0 / 2 \).

Consider a one-dimensional, infinitely long Josephson junction parallel to the \( x \) axis and assume that the critical current density \( j_c(x) \) is a random function taking positive and negative values. We denote the typical length scale of \( j_c(x) \) as \( l \) and define the average value of the critical current density \( \langle j_c \rangle \) as

\[
\langle j_c \rangle = \frac{1}{L} \int_0^L j_c(x) dx,
\]

with the averaging interval \( L \gg l \). The effective Josephson penetration depth \( \Lambda_j \) is introduced by

\[
\Lambda_j^2 = \frac{c \Phi_0}{16 \pi^2 \lambda \langle j_c \rangle},
\]

where \( \lambda \) is the London penetration depth. In the case of a Josephson junction with \( \lambda \ll l \ll \Lambda_j \) the phase difference \( \varphi(x) \) satisfies the equation\(^4\)

\[
\Lambda_j^2 \varphi'' = \frac{j_c(x)}{\langle j_c \rangle} \sin \varphi = 0.
\]

We write the critical current density \( j_c(x) \) as

\[
j_c(x) = \langle j_c \rangle [1 + g(x)],
\]

with \( \langle g(x) \rangle = 0 \). The length scale of variation of \( g(x) \) is of the order of \( l \) and the typical values of \( \max|g(x)| \) vary from \( \max|g(x)| \approx 1 \) to \( \max|g(x)| \gg 1 \). In particular, if the average \( \langle j_c \rangle \) is small compared to the typical amplitude of \( j_c(x) \), then \( \max|g(x)| \gg 1 \).
In terms of the function \( g(x) \) we rewrite Eq. (3) for the phase difference \( \varphi(x) \) in the form
\[
\Lambda_j^2 \varphi'' - [1 + g(x)] \sin \varphi = 0.
\] (5)

The idea of the following calculations is based on the assumption that \( l \ll \Lambda_j \) and on a mechanical analogy. Namely, Eq. (5) for the phase difference \( \varphi(x) \) coincides with the equation describing the motion of a pendulum with a vibrating pivoting point (Kapitza pendulum). In this mechanical analogy \( \varphi \) is the angle determining the position of the pendulum and \( x \) is the time. We treat the case when \( g(x) \) is rapidly alternating over the length of \( \Lambda_j \). In terms of the mechanical analogy, this means that the dependence of the phase difference \( \varphi \) on \( x \) is similar to the trajectory of a pendulum with a rapidly vibrating pivoting point. In this case, the motion of the pendulum is a slow motion along a certain smooth trajectory with rapid oscillations around this trajectory; the amplitude of these oscillations is small. Therefore, we write the phase difference \( \varphi(x) \) as
\[
\varphi(x) = \psi(x) + \xi(x),
\] (6)
where \( \psi(x) \) is a smooth function with length scale of order \( \Lambda_j \) and \( \xi(x) \) is a rapidly alternating function with length scale of order \( l \). The average value of \( \xi(x) \) vanishes and the typical amplitude of variations of \( \xi(x) \) is small, i.e., \( \langle \xi(x) \rangle = 0 \) and \( \langle |\xi(x)| \rangle \ll 1 \).

Substituting the ansatz (6) into Eq. (5) and keeping terms up to first order in \( \xi(x) \) we find
\[
\Lambda_j^2 \psi'' + \Lambda_j^2 \xi'' - [1 + g(x)] [\sin \psi + \xi \cos \psi] = 0.
\] (7)
Two types of terms appear in Eq. (7): terms alternating over a length \( l \) and smooth terms varying over a length \( \Lambda_j \). The alternating terms cancel each other, independently of the smooth terms, which also cancel each other. As a result we obtain both functions \( \psi(x) \) and \( \xi(x) \) from one equation (7).5

First, the alternating phase \( \xi(x) \) is determined by
\[
\Lambda_j^2 \xi'' = g(x) \sin \psi.
\] (8)
In the derivation of Eq. (8) we omitted two out of three alternating terms in Eq. (7) since they are proportional to \( \xi(x) \) and therefore much smaller than \( g(x) \).

Second, an equation for the phase \( \psi(x) \) is derived by averaging Eq. (7) over lengths \( a \gg l \) \( (a \ll \Lambda_j) \), yielding
\[
\Lambda_j^2 \psi'' - \sin \psi - \langle g(x) \xi(x) \rangle \cos \psi = 0.
\] (9)

Introducing the Fourier transform of \( g(x) \) by
\[
g(k) = \frac{1}{2\pi} \int^{\infty}_{-\infty} g(x) e^{ikx} dx,
\] (10)
we find the solution of Eq. (8) in the form of
\[
\xi(x) = \frac{\sin \psi}{2\pi \Lambda_j^2} \int^{\infty}_{-\infty} g(k) e^{ikx} dk = -\xi_g(x) \sin \psi.
\] (11)
The function \( \xi_g(x) \) depends only on the alternating component of the critical current density \( \langle j_c \rangle g(x) \) and the self-generated flux \( \Phi_{g}(x) \), a measurable quantity, is
\[
\Phi_{g} = \Phi_{0} \frac{\xi}{2\pi} = -\Phi_{0} \frac{\xi_{g}}{2\pi} \sin \psi.
\] (12)

This relation allows one to study \( \xi_{g}(x) \) experimentally. In deriving Eq. (11) we ignored the dependence of \( \sin \psi \) on \( x \). This may be done since on the length scale \( l \) the variations of the smooth function \( \sin \psi(x) \) are of order \( l/\Lambda_j \ll 1 \). Taking into account that the alternating part of the critical current density has typical wave numbers \( k \sim 1/l \) we estimate \( \xi_{g}(x) \) from Eq. (11) as
\[
\xi_{g}(x) \sim -\frac{l^2}{\Lambda_j^2} g(x).
\] (13)

Therefore, typical values of the alternating phase difference \( \xi(x) \) are small \( \langle |\xi(x)| \rangle \ll 1 \) if
\[
\langle |g(x)| \rangle \ll \frac{\Lambda_j^2}{l^2}.
\] (14)

Next, using Eqs. (8) and (11), we calculate the average \( \langle g(x) \xi(x) \rangle \) and thus obtain the equation (9) for the smooth phase shift \( \psi(x) \) in the final form:
\[
\Lambda_j^2 \psi'' = \sin \psi - \gamma \sin \psi \cos \psi = 0,
\] (15)
where the constant \( \gamma = \langle g(x) \xi_g(x) \rangle \) is given by
\[
\gamma = \frac{c \Lambda_j}{\Phi_{0} \langle j_c \rangle} \left( \frac{B_{s}^2}{B_{J}^2} \right) = \left( \frac{B_{s}^2}{B_{J}^2} \right).
\] (16)

Here,
\[
B_{s} = \frac{4\pi}{c} \langle j_c \rangle \int g(x) dx = \frac{\Phi_{0}}{4\pi \Lambda_j} \frac{d \xi_{g}}{dx},
\] (17)
is the magnetic field generated by the alternating component of the critical current \( \langle j_c \rangle = 0 \), and
\[
B_{J} = \frac{4\pi}{c} \langle j_c \rangle \Lambda_j.
\] (18)

We estimate the value of \( \gamma \) as
\[
\gamma \sim \frac{l^2}{\Lambda_j^2} \langle g(x) \rangle^2.
\] (19)

The assumption \( \langle |\xi(x)| \rangle \ll 1 \) restricts the value of \( \gamma \). However, it follows from Eqs. (14) and (19) that \( \langle |\xi(x)| \rangle \ll 1 \) and \( \gamma > 1 \) hold simultaneously if
\[
\frac{\Lambda_j}{l} \ll \langle |g(x)| \rangle \ll \frac{\Lambda_j^2}{l^2}.
\] (20)

In particular, Eq. (20) is satisfied when the average value of the critical current density is small compared to the typical amplitude of its variations, i.e., \( \langle j_c \rangle \ll \max |j_c(x)| \).

To complete the description of a stationary state with a self-generated flux we calculate the energy of a Josephson junction \( E \). It takes the form \( E = E_{0} + E_{\xi} \), where \( E_{0} \) is independent of the phase difference \( \varphi(x) \) and
Using Eqs. (6), (11), and (16), we obtain the energy \( E_\phi \) in terms of the smooth phase shift \( \psi(x) \), namely,

\[
E_\phi = \frac{\hbar}{2e} \int dx \left\{ \frac{1}{2} \Lambda_f^2 \phi'^2 - \frac{1}{2} \left[ 1 + g(x) \right] \cos \phi \right\}.
\]

The solutions \( \psi(x) \) of Eq. (15) correspond to the minima and maxima of the functional \( E_\phi[\psi(x)] \) [Eq. (22)].

Let us now apply Eqs. (15) and (22) to two stationary states in a Josephson junction with length \( L \gg \Lambda_f \). The first state occurs in zero applied field. In this case the average flux inside the junction is zero and thus an alternating self-generated flux \( \Phi_s(x) \) appears simultaneously with a phase shift \( \phi = \text{const} \). The second state occurs in nonzero applied field. In this case the average flux inside the junction is carried by Josephson vortices. The alternating self-generated flux \( \Phi_s(x) \) appears then simultaneously with a phase shift \( \phi(x) \), whose spatial dependence describes the vortices.

In the stationary state with \( \phi = \text{const} \) the values of \( \phi \) are determined by Eq. (15) which takes the form

\[
\sin \phi(1 - \gamma \cos \phi) = 0.
\]

In the case \( \gamma \ll 1 \) Eq. (23) has two solutions, \( \psi = 0 \) and \( \psi = \pi \) and thus, cf. Eq. (12), there is no self-generated flux.

In the case \( \gamma > 1 \) there are four solutions of Eq. (23), namely, \( \psi = -\psi_y, 0, \psi_y, \pi \), where

\[
\psi_y = \arccos(1/\gamma).
\]

The energy of a Josephson junction \( E \) has a minimum for \( \psi = \pm \psi_y \) and a maximum for \( \psi = 0, \pi \). It follows then from Eqs. (12) and (24) that a self-generated flux

\[
\Phi_s(x) = -\Phi_0 \frac{\xi_0(x)}{2 \pi} \sin \psi_y = \pm \Phi_0 \frac{\xi_0(x)}{2 \pi} \sqrt{\frac{\gamma^2 - 1}{\gamma}}
\]

arises in the two states with the minimum energy \( E \). Note that the criterion \( \gamma > 1 \) means that the average energy density of the self-generated magnetic field \( \langle B^2 \rangle /4\pi \) is higher than \( \Phi_0^2/4\pi c \lambda \).

From Eqs. (13) and (25) we estimate \( |\Phi_s(x)| \) as

\[
|\Phi_s(x)| \approx -\Phi_0 \sqrt{\frac{\gamma^2 - 1}{\gamma}} \frac{\xi_0(x)}{2 \pi} g(x) < \Phi_0.
\]

Note also that the average tunnel current density \( \langle j \rangle \) is equal to zero for the state with \( \psi = \text{const} \) as

\[
\langle j \rangle = \langle j_s(x) \sin \phi \rangle = \langle j_s \rangle \sin \phi(1 - \gamma \cos \phi).
\]

Next we consider a Josephson vortex. In this case the phase \( \psi(x) \) satisfies Eq. (15) and the boundary conditions \( \psi'(+\infty) = 0 \). It is convenient to write Eq. (15) as

\[
\Lambda_f^2 \psi'' = -\frac{dU}{d\psi},
\]

where the function \( U(\psi) \) is given by

\[
U(\psi) = -1 + \cos \psi + \frac{\gamma}{2} \sin^2 \psi.
\]
can be found if \(0 < \gamma - 1 < 1\). In this case, \(\psi^2 \sim 2(\gamma - 1)\) and \(U(\psi) \sim \psi^2 + 4 - \psi^4 / 8\). It follows then from Eq. (28) that

\[
\psi_1(x) = \sqrt{2(\gamma - 1)} \tanh \left( \frac{x}{\Lambda_J} \sqrt{\frac{\gamma - 1}{2}} \right),
\]

i.e., the flux \(\Phi_1 = \sqrt{2(\gamma - 1)} / \pi \Phi_0\) is localized over a length \(\Lambda_J / \sqrt{\gamma - 1} \gg \Lambda_J\), and the energy is

\[
\varepsilon_1 = \frac{2 \sqrt{2}}{3} \frac{h(j_c) \Lambda_J}{e} (\gamma - 1)^{3/2}.
\]

In the case of \(\gamma \gg 1\) we have \(\psi_c \approx \pi / 2\) and therefore \(\Phi_1 \approx \Phi_2 \approx \Phi_0 / 2\). The main contribution to the potential \(U(\psi)\) (21) comes from the term \((\gamma / 2) \sin^2 \psi\) and both the first and second vortices are described by

\[
\psi_{1,2}(x) = \arcsin \left[ \tanh \left( \frac{x}{\Lambda_J} \sqrt{\gamma} \right) \right].
\]

Each of the two vortices is localized in a region of size \(\Lambda_J / \sqrt{\gamma} = \Phi_0 / (4\pi \Lambda \sqrt{\langle B^2 \rangle}) \ll \Lambda_J\) and has an energy \(\varepsilon_{1,2}\) which is independent of \(\langle j_c \rangle\),

\[
\varepsilon_{1,2} = \frac{h(j_c) \Lambda_J}{e} \sqrt{\gamma} = \frac{\Phi_0}{4\pi^2 \sqrt{\langle B^2 \rangle}}.
\]

It follows from Eq. (12) that a self-generated flux exists in the presence of a Josephson vortex for any value of the parameter \(\gamma\) since \(\psi \neq 0\). The alternating flux \(\Phi_s(x)\) is then proportional to \(\xi_s(x) \sin \psi(x)\), i.e., the rapidly alternating component \(\xi_s(x)\) is modulated by a smooth factor \(\sin \psi(x)\) imposed by the vortex. In the case \(\gamma < 1\) a state with a self-generated flux can be studied experimentally in the presence of vortices.

Let us now illustrate our calculations by the model \(j_c(x) = \langle j_c \rangle + j_c \sin(2\pi x / l)\). From Eqs. (11) and (16) we find

\[
\xi_s(x) = \left( 4l^2 f_c / \Phi_0 \right) \sin(2\pi x / l) / e \Phi_0 \langle j_c \rangle.
\]

As expected \(\xi_s(x)\) depends only on the alternating component of \(j_c(x)\) and a self-generated flux exists (\(\gamma > 1\)) if the amplitude is \(j_c > \langle j_c \rangle\).

In conclusion, we have shown that in a long Josephson junction with an alternating critical current density, a self-generated flux arises if the average energy density of the self-generated magnetic field is higher than a critical value proportional to the average critical current density. Two types of Josephson vortices exist under these conditions: one with flux \(\Phi_1 \ll \Phi_0 / 2\) and one with the complementary flux \(\Phi_2 = \Phi_0 - \Phi_1 > \Phi_0 / 2\). The stability of these vortices remains to be checked.

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3. An alternative explanation suggested by M. B. Walker, Phys. Rev. B 54, 13 269 (1996) attributes the self-generated flux to vortices with a flux of \(\Phi_0 / 2\) trapped at points where a twin boundary intersects a grain boundary.