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The flux jump and critical state stability in superconductors

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Abstract. The critical state stability in a 'hard' type-II superconductor with respect to the flux jumps has been studied both for adiabatic and isothermal boundary conditions. The sample geometry as well as initial magnetic field and current distributions are found to be important for the stability criterion. The increase of the ratio of the thermal and magnetic diffusivities results in an increase of the stability. This effect is most noticeable in the case of intensive external cooling. For the extremely 'hard' superconductors the simplified scheme of the investigation is proposed.

1. Introduction

In hard superconductors placed in an external magnetic field, inherent instabilities—the so-called flux jumps—are commonly observed (see for example Saint-James *et al* 1969). The fast penetration of some quantized flux lines—Abrikosov vortexes (Abrikosov 1957)—is responsible for this phenomenon. The flux jump can be initiated either by temperature or magnetic field fluctuations. The flux jump nucleation and the possible ways of stabilization of the superconducting devices have been discussed in many papers. The qualitative theory of this for different physical situations have been developed by Wipf (1967), Swartz and Bean (1968) and Wilson *et al* (1970).

In general, one has to investigate the stability of the Maxwell equations combined with the thermal diffusion equation with respect to small perturbations (Kremlev 1973, 1974). Of course, it is necessary to choose a definite model of the critical state.

Wipf (1967), Swartz and Bean (1968) and Wilson *et al* (1970) have determined the stability criterion, disregarding the heat conductivity. This assumption allows the calculations to be reduced considerably. Naturally, the area of applicability of this approximation could not be investigated.

Kremlev (1973) has considered the stability of the flat plate of a finite thickness and the transport current has been equal to zero. The superconductor was assumed to be extremely 'hard', i.e. $D_t/D_m=0$, where

$$D_m=\frac{\rho_f c^2}{4\pi} \tag{1}$$

is the magnetic diffusivity and

$$D_t=\kappa/\nu \tag{2}$$

is the heat diffusivity.

Here ρ_f is the normal state resistivity, κ is the heat conductivity and ν is the heat capacity of the superconductor; c is the velocity of light. In this approximation the current density always keeps pace with the temperature and is at any moment equal to the critical value for the given temperature. In addition, it was assumed that $\partial j_c/\partial B=0$ where j_c is the critical current density and B is the magnetic field induction within the sample. This model of the superconducting critical state is usually referred to as Bean's (1974) model.

Koyama (1973) has taken into account heat conductivity, but for the sake of simplicity, reduced the order of the differential equations to be investigated. Unfortunately, this procedure, as it has been carried out by Koyama, seems to be inadequate. Some assumptions and the boundary conditions formulated there without proof are incorrect And thus, the results obtained by Koyama (1973) are correct only in some extreme cases.

In the article by Kremlev (1974) the theory has been generalized for an arbitrary ratio D_t/D_m.

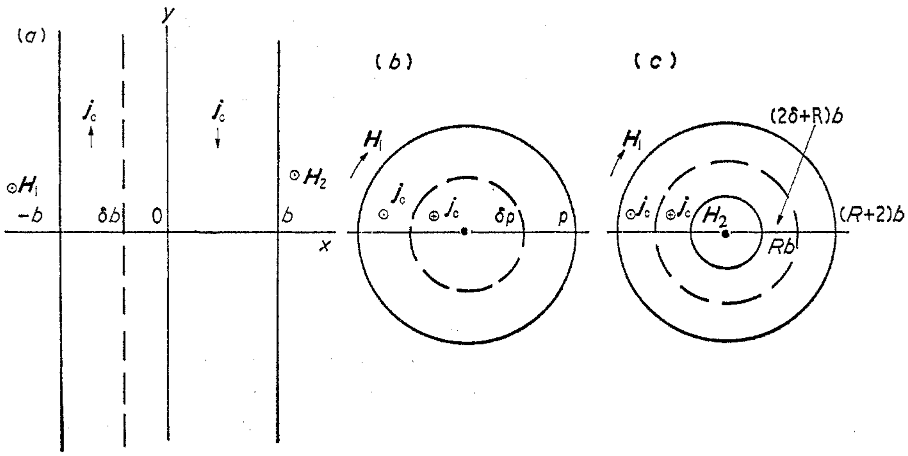

Figure 1. The sample geometry, the magnetic field and current distributions: (a) the flat plate; (b) the wire with a fixed total current; (c) the tube with a fixed total current.

In the present work the problem of the critical state stability of 'hard' superconductors with respect to the flux jumps is discussed in the framework of the theory developed by Kramlev (1973, 1974) on the bases of Bean's (1964) critical state model. The fourth-order equation for a small temperature perturbation (Kremlev 1974) in some special cases may be transformed to obtain two independent second-order differential equations; that considerably simplifies the mathematical procedure. On the basis of these results the stability criterion for the cylindrical samples (figure 1 (b, c)) is found; also the flat plate of finite thickness is considered in the case of non-zero transport current (figure 1(a)). Both adiabatic and isothermal conditions on the external boundaries are discussed. It is shown that the fourth-order system of the differential equations may be reduced to the second-order one in the case of $D_t/D_m \ll 1$. Thus, one has to solve only second-order differential equation to find the stability criterion for the extremely 'hard' superconductors.

2. Basic equations

It is convenient to present here the basic equations derivation (Kremlev 1973, 1974) for the three-dimensional geometry. Let us denote the initial temperature of the superconductor as T_i and let θ be a small temperature perturbation ($\theta \ll T_c - T_i$, where T_c is a critical temperature of the superconductor). The thermal diffusion equation may be written (to the linear approximation for small θ) as

$$\nu \frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta + \boldsymbol{j}_c \boldsymbol{E} \tag{3}$$

where $j_c = j_c(T_i)$ and E is the electric field which appeared during vortex lines motion. We may write the relation between the current density j and the electric field E in the form

$$\boldsymbol{j} = \boldsymbol{j}_c(T) + \rho_f^{-1} \boldsymbol{E} \tag{4}$$

or to the first approximation with respect to θ

$$\boldsymbol{j} = \boldsymbol{j}_c + \frac{\partial \boldsymbol{j}_c}{\partial T} \theta + \rho_f^{-1} \boldsymbol{E}. \tag{5}$$

The Maxwell equation for E is

$$\text{curl curl } \boldsymbol{E} = -\frac{4\pi}{c^2} \frac{\partial \boldsymbol{j}}{\partial t} \tag{6}$$

(as usual $H = B$).

The linear dependence $j(E)$ (4) takes place only for sufficiently large deviation from the equilibrium. Nevertheless, this is not essential for the stability investigations. Indeed, the nonlinear section of the curve $j(E)$, as a rule, is small and the increased fluctuations would reach the linear region of the curve; decreased fluctuations are not of interest for the present purpose.

The model proposed by Kremlev (1973, 1974), which is used in our work, is true only in the absence of the so-called Bean–Livingstone's surface barrier (see for example Bean and Livingstone 1964, de Gennes 1969), which prevents the free flux propagation through the surface. In particular, this means, that external magnetic field must be sufficiently large.

In the Bean model $\partial j_c/\partial T$ may be expressed as

$$\frac{\partial j_c}{\partial T} = -\frac{j_c}{T_0} \qquad T_0 = T_0(T_i) \tag{7}$$

and eliminating E and j from (3)–(6) we find

$$\nabla^2 \left[\boldsymbol{e} \left(\nu \frac{\partial \theta}{\partial t} - \kappa \nabla^2 \theta \right) \right] = \boldsymbol{e} \left[-\frac{4\pi}{c^2} \left(\frac{j_c^2}{T_0} \frac{\partial \theta}{\partial t} - \frac{\nu}{\rho_f} \frac{\partial^2 \theta}{\partial t^2} + \frac{\kappa}{\rho_f} \frac{\partial}{\partial t} \nabla^2 \theta \right) \right] \tag{8}$$

where $\boldsymbol{e}$ is a unit vector in the E direction. One may express θ in the form

$$\theta = \exp\left(\lambda t \frac{\kappa}{\nu L^2} \right) \chi(\boldsymbol{x}/L) \tag{9}$$

where L is some length typical for each sample. If in the cylindrical coordinates r, z, ϕ the electric field E has only a z component and the perturbation is independent of the coordinate z, then the substitution (9) to (8) gives the following equation for χ:

$$\nabla^2(\nabla^2 \chi) - \lambda(1+\tau) \nabla^2 \chi - \lambda(\beta - \lambda\tau) \chi = 0 \tag{10}$$

where

$$\beta = \frac{4\pi}{c^2} \frac{j_c^2 L^2}{\nu T_0} \tag{11}$$

$$\tau = \frac{4\pi}{c^2} \frac{\kappa}{\nu \rho_f} = \frac{D_t}{D_m}. \tag{12}$$

It follows from (10), that to the zero-order approximation (for $D_t/D_m \ll 1$)

$$\nabla^2(\nabla^2\chi) - \lambda\nabla^2\chi - \lambda\beta\chi = 0. \tag{13}$$

We should emphasize that the transition from (10) to (13) corresponds to the limit $\rho_f = \infty$ and not to $\kappa = 0$, since in the latter limit the expression (9) for the solution θ is invalid.

Thus the critical state stability investigation is reduced to the definition of parameters β and τ for which λ changes its sign. One has a stable system for negative λ and unstable for positive λ. The stability criterion should obviously depend upon thermal and electrical boundary conditions.

Denoting

$$\begin{aligned} \lambda_\tau &= \lambda(1+\tau) \\ \beta_\tau &= \frac{1}{1+\tau}(\beta - \lambda\tau) \end{aligned} \tag{14}$$

one may transform (10) to (13) and consider only the latter equation, then, using formulae (14) and taking into account necessary modifications of boundary conditions, generalize the results for arbitrary τ (i.e. arbitrary D_t/D_m).

Equation (13) may be transformed to the more simple form. First, (13) is equivalent to the system

$$\left.\begin{aligned} \nabla^2\chi &= \psi \\ \nabla^2\psi &= \lambda\psi + \lambda\beta\chi. \end{aligned}\right\} \tag{15}$$

Substituting into (15) ψ in the form

$$\psi = (\lambda^2/4 + \lambda\beta)^{1/2} f + \frac{\lambda}{2}\chi$$

one can easily obtain

$$\left.\begin{aligned} \nabla^2\chi &= (\lambda/2)\,\chi + (\lambda^2/4 + \lambda\beta)^{1/2} f \\ \nabla^2 f &= (\lambda/2)\, f + (\lambda^2/4 + \lambda\beta)^{1/2}\,\chi. \end{aligned}\right\} \tag{16}$$

Adding and subtracting the equations of the system (16) we have

$$\left.\begin{aligned} \nabla^2 F_1 - k_1^2 F_1 &= 0 \\ \nabla^2 F_2 + k_2^2 F_2 &= 0 \end{aligned}\right\} \tag{17}$$

where

$$F_{1,2} = (\chi \pm f)/2$$

and

$$k_{1,2}^2 = (\lambda^2/4 + \lambda\beta)^{1/2} \pm \lambda/2 \tag{18}$$

$$\chi = F_1 + F_2. \tag{19}$$

Thus, the fourth-order differential equation (13) is reduced to the two independent second-order equations of the standard form (17). For the sample configuration, for which separation of the variables r and ϕ is allowable, to resolve (17) is not too difficult.

3. The stability criterion for the adiabatic case

In this section we shall consider the stability criterion in the absence of the external cooling. The required boundary condition is

$$\nabla\theta=0$$

or

$$\nabla\chi=0. \tag{20}$$

As an electrodynamical boundary condition, we assume that the external magnetic field is constant in time (therefore, the total transport current remains constant during the process). Then, on the surface of the sample

$$\frac{\partial \boldsymbol{H}}{\partial t}=-c\,\mathrm{curl}\,\boldsymbol{E}$$

and from (3) and (9)

$$\mathrm{curl}\,[\boldsymbol{e}\,(\lambda\chi-\nabla^2\chi)]=0 \tag{21}$$

where $\boldsymbol{e}$ is a unit vector in the electric field direction.

Further, the stability investigations should be carried out for the samples with geometry shown in figure 1(*a–c*). In practice, the solution χ is found in two regions because each case and matching condition must be formulated. These two regions differ by the direction of the current, and it is expedient to take the current density on their boundary as equal to zero. Denoting this boundary by δ, we have for the electric field

$$E\,(\delta\pm0)=0$$

or

$$\lambda\chi-\nabla^2\chi\,|_{\delta\pm0}=0. \tag{22}$$

Besides the temperature θ and heat flux $\kappa\nabla\theta$ must be continuous at δ:

$$\left.\begin{aligned}&\theta(\delta+0)=\theta(\delta-0)\\ \text{or}\quad &\chi(\delta+0)=\chi\,(\delta-0)\end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned}&\nabla\theta\,(\delta+0)=\nabla\theta(\delta-0)\\ \text{or}\quad &\nabla\chi(\delta+0)=\nabla\chi(\delta-0).\end{aligned}\right\} \tag{24}$$

Substituting the solution χ of the system (17) into equation (20)–(24) one gets a set of linear equations for arbitrary constants contained in χ. The requirement of the existence of nontrivial solution χ is able to give us the spectrum $\lambda(\beta)$, which depends, of course, on the geometrical parameters R, δ, b (see figure 1). The stability criterion is defined by the minimum value of β for which $\lambda\,(\beta)>0$.

Note that the boundary δ comes into motion during the perturbation process. Its velocity may be readily found from the constant current condition.

Here and in the §4 we shall consider only the case with $\tau=0$. The situation with $\tau\neq0$ will be discussed in §5.

3.1. *Flat plate*

For the sample with geometry shown in figure 1(*a*), the equation (13) has a form (we set L equal to b)

$$\chi''''-\lambda\chi''-\lambda\beta\chi=0 \tag{25}$$

where β is

$$\beta = \frac{4\pi}{c^2} \frac{j_c^2 b^2}{\nu T_0} \tag{26}$$

The boundary conditions (20)–(24) for this case are:

$$\left.\begin{aligned} &\chi'(\pm 1)=0;\ \lambda\chi'(\pm 1)-\chi'''(\pm 1)=0 \\ &\chi(\delta+0)=\chi(\delta-0);\ \chi'(\delta+0)=\chi'(\delta-0) \\ &\lambda\chi(\delta\pm 0)-\chi''(\delta\pm 0)=0. \end{aligned}\right\} \tag{27}$$

The equation (25) with the relations (27) has a nontrivial solution if

$$\kappa_2^3\{\tan[k_2(1+\delta)]+\tan[k_2(1-\delta)]\}=k_1^3\{\tanh[k_1(1+\delta)]+\tanh[k_1(1-\delta)]\} \tag{28}$$

where k_1 and k_2 are defined by (18). For $\delta=0$ the equation (28) gives a relation obtained

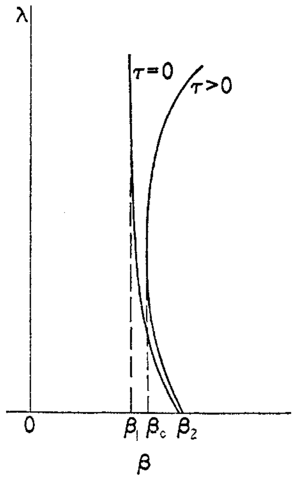

Figure 2. The qualitative behaviour of the function $\lambda(\beta)$ for adiabatic boundary conditions at various τ: $\beta_1=\beta(\lambda=\infty)$, $\beta_2=\beta(\lambda=0)$.

by Kremlev (1973). The qualitative behaviour of the curve $\lambda(\beta)$ is shown in figure 2. Denoting

$$\beta_1=\beta(\lambda=\infty) \quad \text{and} \quad \beta_2=\beta(\lambda=0) \tag{29}$$

one may easily find

$$\begin{aligned} &\beta_1=\pi^2/(4(1+|\delta|)^2) \\ &\beta_2=3/(1+3\delta^2). \end{aligned} \tag{30}$$

We should note that for $|\delta|=1$ the system (27) has the only solution $\chi\equiv 0$. Hence $|\delta|=1$ may be understood only as a limit $|\delta|\to 1$.

The dependences $\beta_1(\delta)$ and $\beta_2(\delta)$ are plotted in figure 3. The stability is defined by the parameter β_1. In our terms the total transport current is expressed as

$$I=2\delta b j_c. \tag{31}$$

For $\delta=0 (I=0)$ the stability parameter has a maximum and the expression (30) is equivalent to the results obtained by the previous authors. For $|\delta|=1$ ($I=I_{\max}$) the stability criterion decreases by a factor of four with respect to the maximum.

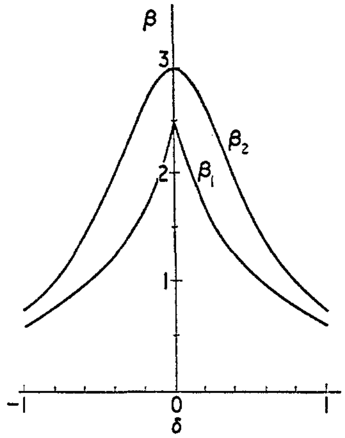

Figure 3. The dependences $\beta_1(\delta)$ and $\beta_2(\delta)$ for the flat-plate, $\tau=0$, adiabatic conditions.

3.2. The cylindrical samples with a given transport current

For the samples with configurations shown in figure 1(*b*) and (*c*) the system (17) has a form

$$\left.\begin{aligned}&\frac{d^2F_1}{dr^2}+\frac{1}{r}\frac{dF_1}{dr}-k_1^2F_1=0\\&\frac{d^2F_2}{dr^2}+\frac{1}{r}\frac{dF_2}{dr}+k_2^2F_2=0\\&\chi=F_1+F_2\end{aligned}\right\}. \tag{32}$$

The solution of each equation (32) is the zero-order Bessel function. It is convenient to write χ in the form

$$\chi=c_1J_0(k_2r)+c_2N_0(k_2r)+c_3I_0(k_1r)+c_4K_0(k_1r) \tag{33}$$

where J_0 and N_0 are Bessel functions of the first and second kind, and I_0 and K_0 are modified Bessel functions of the first and second kind.

For the sample geometry shown in figure 1(*c*) (the tube with a fixed transport current) the typical length L we set equal to b—half a thickness of the tube wall. Then β is determined by the expression (26). The required boundary conditions are

$$\frac{d\chi}{dr}=0;\ \frac{d}{dr}\left[\lambda\chi-\frac{1}{r}\left(r\frac{d\chi}{dr}\right)\right]=0 \text{ at } r=R \text{ and at } r=R+2$$

and for the continuity of χ and χ' at $r=R+2\delta$

$$\lambda\chi-\frac{1}{r}\frac{d}{dr}\left(r\frac{d\chi}{dr}\right)=0 \text{ at } r=R+2\delta\pm0. \tag{34}$$

The dependence $\lambda(\beta)$ obtained with the help of (33) and (34) has the same form as in the flat case (see figure 2).

For the sample geometry shown in figure 1(*b*), the wire with a fixed current, we shall define L as $\rho/2$; then β is determined by

$$\beta=\frac{4\pi}{c^2}\frac{j_c^2}{\nu T_0}\frac{\rho^2}{4} \tag{35}$$

and the boundary conditions have a form

$$\frac{d\chi}{dr}=0;\ \frac{d}{dr}\left[\lambda\chi-\frac{1}{r}\frac{d}{dr}\left(r\frac{d\chi}{dr}\right)\right]=0 \text{ at } r=2 \tag{36}$$

and for the continuity χ and χ' at $r=2\delta$

$$\lambda\chi-\frac{1}{r}\frac{d}{dr}\left(r\frac{d\chi}{dr}\right)=0 \text{ at } r=2\delta\pm 0.$$

The regularity of χ at $r=0$ requires that c_2 and c_4 in (33) be equal to zero in the region $r<2\delta$. The expressions $\lambda(\beta)$ for the wire and for the tube with $R=0$ coincide with each other. The stability criterion is defined by the parameter β_1 as in the case of the plate.

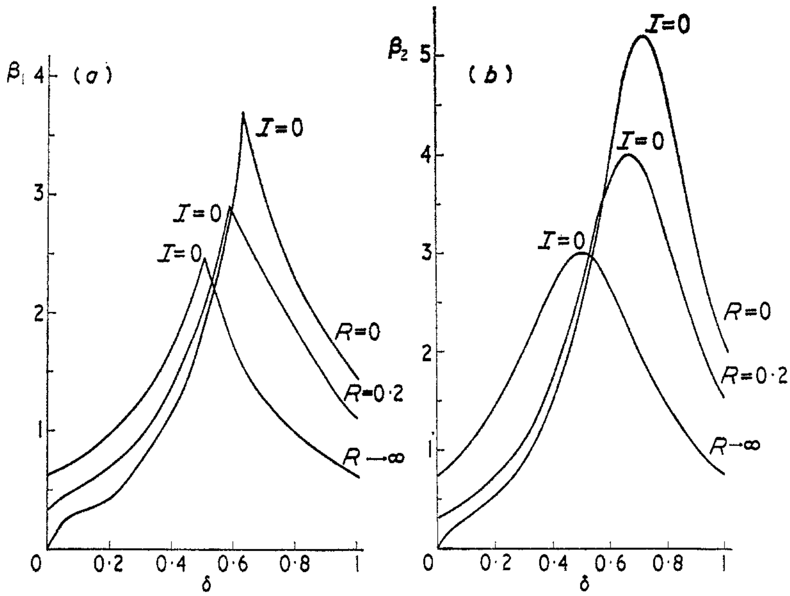

Figure 4. The values (*a*) $\beta_1(\delta)$ and (*b*) $\beta_2(\delta)$ for the cylindrical samples with different R, $\tau=0$, adiabatic conditions.

The plots of $\beta_1(\delta)$ and $\beta_2(\delta)$ are given in figure 4(*a*) and (*b*). One may consider $\delta=0$ and $\delta=1$ only as the corresponding limits $\delta\to 0$ and $\delta+1$. The total transport current is

$$I=4\pi b^2 j_c\,[1-2\delta^2+R(1-2\delta)]. \tag{37}$$

The maximum stability takes place in the state with $I\neq 0$ in the contrast with the flat geometry; e.g. $I(\beta_{max})\simeq\pi b^2 j_c$ for $R=0$.

We should emphasize that, for the equal values of transport current, the stability depends upon the magnetic field distribution and hence upon the magnetic history of the sample. This dependence is the most striking at small R. For $\delta\simeq 0$ and $\delta\simeq 1$, with a difference in the magnetic field distribution only (see figure 5(*a*) and (*b*)), the stability parameter β_1 changes at $R=0$ from $\beta_1=0$ to $\beta_1\simeq 1{\cdot}5$, although the transport currents are equal in both cases. This dependence disappears with an increase of R. Naturally, at $R\to\infty$ we get the stability criterion for the flat plate.

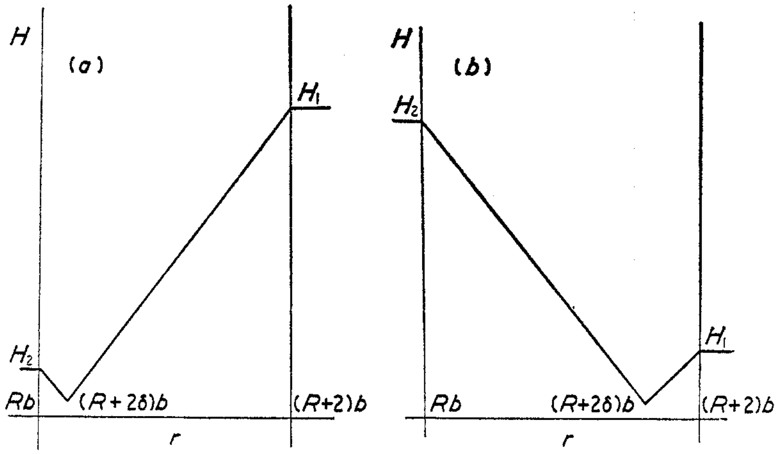

Figure 5. The distribution of $H(r)$ for the cylindrical superconductor: (*a*) $\delta \simeq 0$; (*b*) $\delta \simeq 1$.

4. Isothermal boundary conditions

We shall discuss in this section the critical state stability of the superconductor with intensively cooled surface. The obvious transformation has to be performed only in the thermal boundary conditions: the adiabatic condition (20) must be changed to the isothermal one:

$$\theta = 0$$

or

$$\chi = 0. \tag{38}$$

In equations (27), (34), (36) obtained for the samples shown in figure 1, the evident transformation in each case takes place only for the first equation.

4.1. Flat plates

For the sample geometry shown in figure 1(*a*) the function $\lambda(\beta)$ is plotted in figure 6. The stability criterion is defined by β_1, and is just the same as in the adiabatic case. The disturbances with $\lambda = 0$ are completely damped and β_2 goes to infinity, since in this case the rate of cooling is larger than the rate of heating due to flux motion. For the subsequent discussion it is useful to obtain the value $\lambda(\beta)$ for $\lambda \gg 1$. One may easily find in the case under consideration that

$$\lambda \sim \frac{\pi^2}{16\,(1+|\delta|)^4} \times \frac{1}{\beta - \pi^2/[4(1+|\delta|)^2]} \tag{39}$$

4.2. Cylindrical samples

For samples with a cylindrical symmetry (figure 1*b*, *c*) the dependence $\lambda(\beta)$ has an appearance similar to the curve shown in figure 6. The stability parameter β_1 remains the same as in the adiabatic case. The function $\lambda(\beta)$ for $\lambda \gg 1$ may be easily found, but the respective algebraic expression is too bulky to be presented in the text.

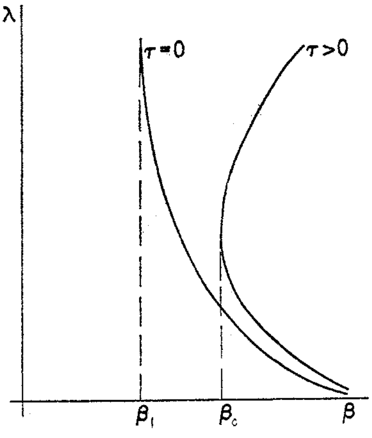

Figure 6. The function $\lambda(\beta)$ for the isothermal case at different τ; $\beta_1=\beta(\lambda=\infty)$.

5. The stability criterion for finite D_m

In this section we shall omit the limitation $\tau=0$ (i.e. $D_m \rightarrow \infty$) and hence (see §2) take into account the normal current density induced in the resistive state. Physically, this gives rise to the appearance of the viscous forces. They prevent the flux lines from the motion through the bulk of superconductor; that results in the increase of the stability with respect to the perturbations with large λ (because of the increase of the viscous force with velocity).

As was mentioned in §2, the equation (10) for an arbitrary τ may be reduced to (13) (with $\tau=0$) by the transformation (14). Unfortunately, the electrodynamical boundary condition (21) and (22) contains the parameter λ directly.

With the help of (14) we have

$$\operatorname{curl}\left[e\left(\frac{\lambda\tau}{1+\tau}\chi-\nabla^2\chi\right)\right]=0$$

$$\frac{\lambda\tau}{1+\tau}\chi-\nabla^2\chi=0 \tag{40}$$

and the stability investigations for $\tau\neq 0$ have to be performed again. The dependence $\lambda_\tau(\beta_\tau)$ is to be found from the equation (13) with boundary conditions (20) or (38) (40), (23) and (24), then by means of (14) one readily obtains the value $\lambda(\beta, \tau)$. The qualitative behaviour of the curve $\lambda(\beta)$ for $\tau\neq 0$ in both adiabatic and isothermal cases is shown in figure 2 and 6. The new stability parameter β_c is determined by the condition (see figures 2 and 6)

$$\left(\frac{d\lambda}{d\beta}\right)_{\beta_c}=\infty \tag{41}$$

or, from (14),

$$\left(\frac{d\lambda_\tau}{d\beta_\tau}\right)\beta_{\tau c}=-\frac{(1+\tau)^2}{\tau}. \tag{42}$$

The procedure described above may be considerably simplified in the case $\tau\ll 1$.

This assumption is valid for the majority of the 'hard' superconductors. At this limit the relation (14) becomes

$$\lambda = \lambda_\tau$$
$$\beta = \beta_\tau + \lambda_\tau \tau. \tag{43}$$

The term $\lambda_\tau \tau$ in the second of the equations (43) cannot be disregarded if the equilibrium is broken down by the disturbances with large λ. In this approach the relations (40) and (21), (22) are equivalent. Therefore

$$\lambda_\tau(\beta_\tau) = \lambda(\beta, \tau = 0) \tag{44}$$

and (42) has a form

$$\left(\frac{d\lambda_\tau}{d\beta_\tau}\right)\beta_{\tau c} = -\frac{1}{\tau};\ \beta_c = \beta_{\tau c} + \lambda_{\tau c}\tau. \tag{45}$$

Thus the function $\lambda(\beta, \tau)$ can be obtained to the first approximation by means of the results for $\tau = 0$ (§§3 and 4) and the relations (44).

It is easy to see that $\beta_1(\tau=0) < \beta_c < \beta_2(\tau=0)$ for $\tau \ll 1$ and the difference between $\beta_1(\tau=0)$ and $\beta_2(\tau=0)$ is not of significance in the adiabatic case (§3). Hence, the stability is not considerably affected by τ. In isothermal case these speculations are not true and the value of τ may be important (see also Kremlev 1974).

For the flat plate (figure 1(*a*)) with the help of (39) and (45) one may find

$$\beta_c \simeq \frac{\pi^2}{4(1+|\delta|)^2}(1+2\tau^{1/2}). \tag{46}$$

The value $\beta_c(\tau)$ for different δ is shown in figure 7(*a*), and β_c increases approximately by the factor 1·5 as τ changes from 0 to 0·1. The situation similar to the latter one takes

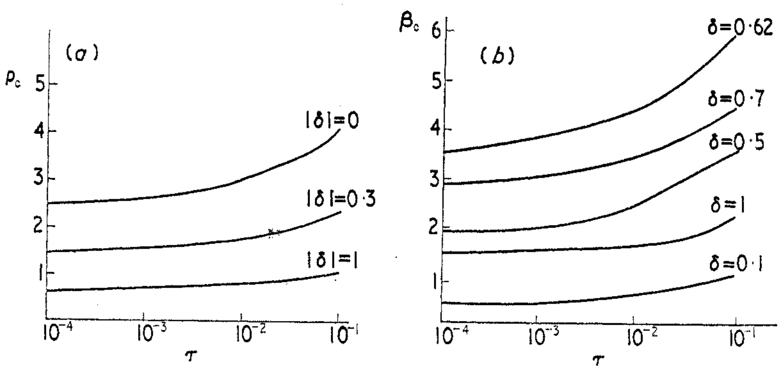

Figure 7. The dependence $\beta_c(\tau)$ for various δ in the isothermal case: (*a*) flat plate; (*b*) cylindrical sample with $R=0$.

place for the cylinder with $R=0$ (see figure 7(*b*)). As for $\tau=0$, there is a considerable dependence of the stability criterion upon the magnetic field distribution:

$$\beta_c \simeq \begin{cases} 0 & \delta \simeq 0 \\ 1{\cdot}5-2 & \delta \simeq 1. \end{cases}$$

6. Simplified scheme

As was outlined previously for the samples shown in figure 1, the disturbances with $\lambda \to \infty$ are responsible for the magnetic instabilities in the case $\tau = 0$ (i.e. $\rho_f \to \infty$) both under the isothermal and adiabatic conditions. It seems natural to assume this result to be independent of the sample geometry. Therefore, it is worthwhile proceeding directly to the limit in the general equations. This allows one to simplify the calculation procedure.

The following speculations clarify qualitatively the physical sense of the limit $\lambda \to \infty$. The characteristic time of the heat propagation throughout the sample is

$$t_\kappa \sim \frac{L^2 \nu}{\kappa}$$

and the building-up time of the temperature fluctuation is

$$t_\theta \sim \frac{L^2 \nu}{\lambda \kappa}.$$

For $\lambda \to \infty$

$$\tau_\theta / t_\kappa \sim 1/\lambda \ll 1$$

i.e. the heat exchange cannot occur during the process. Then, one may derive from (3), (5), (6) for $\rho_f \to \infty$

$$\frac{\partial \theta}{\partial t} = \frac{1}{\nu} j_c \cdot \boldsymbol{E} \tag{47}$$

$$\text{curl curl } \boldsymbol{E} = \frac{4\pi}{c^2} \frac{j_c^2}{\nu T_0} \boldsymbol{E}. \tag{48}$$

As the heat flux within the sample is disregarded in this scheme, the thermal boundary conditions, requiring the continuity of the temperature and its first derivative are to be excluded in the correct formulation of the problem. Therefore, we should take into account only electrodynamical boundary conditions:

$$\left.\begin{array}{l} \text{curl } \boldsymbol{E} = 0 \text{ on the boundaries} \\ \boldsymbol{E}(\delta \pm 0) = 0. \end{array}\right\} \tag{49}$$

One may obtain just the same results proceeding to the limit $\lambda \to \infty$ directly in the equations (9), (10) and under the boundary conditions (20) or (38), (21)–(24), but the derivation presented above seems to be more evident.

It follows from (47) that the equilibrium is broken down ($\partial\theta/\partial t > 0$) if there exists a nontrivial solution E of the equation (48) with the conditions (49).

To this approximation, the stability investigations are carried out separately for the regions to the right and to the left side from the boundary δ, as can easily be seen. The last circumstance accounts for the irregularities in the curves $\beta_1(\delta)$ (see figure 4*a*). The equilibrium is broken down by the perturbations with $\lambda \to \infty$ separately at the right or at the left region from δ. The stability of the system as a whole is determined by the less stable zone. For the cylindrical samples (figure 1*b*, *c*) at small δ the external (with respect to δ) region is the more unstable, and at $\delta \sim 1$ the internal region is the more unstable. The point of irregularity corresponds to the change of the most 'dangerous' region.

This method allows us to consider a wide class of the samples for which it is difficult to find the analytical solution of the equation (8).

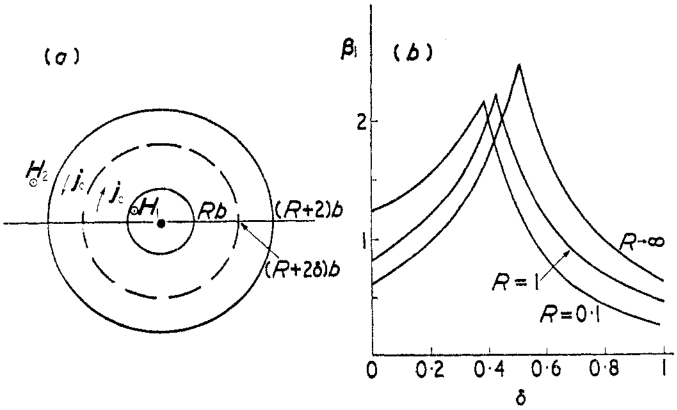

Figure 8. (*a*) The sample configuration, (*b*) parameter $\beta_1(\delta)$ for various R.

For example, in the case of the sample geometry shown in figure 8(*a*) (the tube in the parallel magnetic field), the equation (48) has a form

$$\nabla^2 E + \left(\beta - \frac{1}{r^2}\right) E = 0 \tag{50}$$

where the coordinates are normalized by b (half the thickness of the tube wall), and β is determined by the expression (26). The solution E may be expressed as

$$E = c_1 J_1 (\beta^{1/2} r) + c_2 N_1(\beta^{1/2} r). \tag{51}$$

Substituting (51) to the relations (49) one obtains the condition of the existence of the nontrivial solution $E(r)$ for $r < R+2\delta$ in the form

$$J_1(\beta^{1/2}(R+2\delta))\, N_0(\beta^{1/2} R) - N_1(\beta^{1/2}(R+2\delta))\, J_0(\beta^{1/2} R) = 0 \tag{52}$$

and for $r > R+2\delta$

$$J_1(\beta^{1/2}(R+2\delta))\, N_0(\beta^{1/2}(R+1)) - N_1(\beta^{1/2}(R+2\delta))\, J_0(\beta^{1/2}(R+1)) = 0. \tag{53}$$

The dependence $\beta_1(\delta)$ defined from (52) and (53) is plotted in figure 8(*b*). The transport current normalized by the unit length along the axis may be expressed as $I = 2\delta b j_c$. The stability criterion has a maximum at $I=0$ and depends upon the magnetic history of the specimen, e.g. for $R=0{\cdot}1$, $\beta_1(0) \simeq 1{\cdot}3$ and $\beta_1(1) \simeq 0{\cdot}2$. Naturally, the parameter β_1 defined by the simplified scheme for the samples shown in figure 1 is precisely the same as it was found to be in §§3 and 4.

7. Conclusions

It has been shown that the sample geometry, the initial magnetic field and currents distributions considerably affect the critical state stability in 'hard' superconductors with respect to the flux jumps. For example, in the case of the flat plate (figure 1*a*) the stability decreased with increase of the total transport current (figure 3). For the cylindrical samples there exists a pronounced dependence of the stability parameter upon the initial magnetic field distribution, i.e. the magnetic history of the sample is of importance (figures 4*a* and 8*b*).

It is shown for the extremely 'hard' superconductors that the temperature perturbations with $\lambda \to \infty$ are the most 'dangerous'. This allows us to reduce correctly the order of the system of the differential equations under investigation and to obtain the required number of the boundary conditions. Instead of the fourth-order (with respect to the coordinates) equation (8) it is sufficient to solve the second-order one (48). Naturally, this greatly simplifies all calculations.

The presence of the normal current in the material gives rise to the existence of the viscous force which acts on the flux lines during their motion through the specimen. This force results in the damping of the disturbances with $\lambda \to \infty$ and therefore the stability increases (§5). The variation of the stability parameter may be of importance (figure 7) if the resistance ρ_f of the material in the normal state is not too large (namely $D_t/D_m \gtrsim 0{\cdot}1$), and in the case of effective external cooling (isothermal condition) it allows to increase the stability parameter approximately by the factor 1·5 at $D_t/D_m \sim 0{\cdot}1$.

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