Quantum Oscillations in Superconductors


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Quantum oscillations of the energy gap, current density, and surface impedance in superconductors are investigated. The distinctive features of the oscillations with respect to the magnetic field are elucidated, and the period and temperature dependence of the amplitude are found. The experimental feasibility of establishing the dispersion law ε(p) and the gap ∆(p) from the period and temperature dependence of the amplitude of the oscillations is proved. The conditions on Maxwell’s equations at the interface between metal and vacuum are discussed. The rule for determining the phase γ in the conditions for quasiclassical quantization is found in general form.

1. Introduction

Since Pincus¹ pointed out the existence of surface levels in superconductors, a large number of articles have appeared which are devoted to this question. The possibility of a quasiclassical consideration of such levels was first mentioned in article §2 and the Bohr rules of quantization for the excitations in superconductors are derived. In §3 the quasiclassical treatment was used in order to study vortex lines in a pure superconductor. The asymptotic form of the exact wave functions of the excitations is found in §4 in the quasiclassical domain, and with their aid the values of the phase in the quantization rules are obtained. In §5 it is shown that the number of levels increases rapidly with increasing magnitude of the angle of reflection of the excitation from the surface of the superconductor. An analysis of the nature of the classical trajectories and of the quantized spectrum is carried out in §6; there it was shown that the strong dependence of the number of discrete levels on the magnetic field leads to resonances and to quantum oscillations of the various physical quantities with respect to the magnetic field.¹

The present article is devoted to the calculation of the quantum oscillations of the energy gaps, current density, and surface impedance in a pure superconductor occupying the half-space y ≥ 0 in the presence of a magnetic field H ≡ z. Since the quantum oscillations are essentially connected with the quasiclassical nature of the corresponding levels, we shall discuss this question in somewhat more detail. In the classical case the energy of the excitations in a superconductor in the presence of a magnetic field is given by

\[ ε(p) = ± \sqrt{\frac{\mu^2}{2} + \Delta^2} \pm \frac{\epsilon}{c} A \cdot v \]  

(1.1)

where \( \mu \) is the chemical potential, \( \Delta \) is the energy gap, and \( A \) is the vector potential of the magnetic field.²

According to Eqs. (1.1) and (1.3) the phase trajectories \( p_y = p_y(y) \) associated with the quadratic dispersion law \( ε_e = p^2/2m \) are determined by the equation

\[ p_y = \pm p_y = \{2m(\mu_y + \sqrt{(\epsilon + p)^2 - \Delta^2})\}^{1/2} \]  

(1.4)

\[ \rho(y) = \epsilon v_A v_A / c \]  

(1.5)

The form of the phase trajectory essentially depends on the parameters \( p_{x1}, p_{x2}, \) and \( \epsilon, \) where in general there are four branches \( p_y(y) \) corresponding to each set \( p_{x1}, p_{x2}, \) \( \epsilon, \) in accordance with the four possibilities for the choice of the plus and minus signs in Eq. (1.4). In order to write down the condition for the validity of the quasiclassical treatment, let us turn to the characteristic trajectory depicted in Fig. 1. One can talk about classical motion in that case when a quantum transition from one orbit to the other is excluded. It is clear that the uncertainty in the coordinate is characterized by the attenuation depth \( \delta \) of the field (or by the dimension \( \xi_0 \) of a pair; in our case \( \xi \) will play the fundamental role).

Therefore, the obvious and easily satisfied condition for the absence of transitions from trajectory 1 to trajectory 3 is \( (\delta y) \gg \delta \) (see Fig. 1), where the transitions are absent with exponential accuracy in \( - (\delta y) / \delta \). On the other hand, the uncertainty in the momentum \( p_y \) is given by \( (\delta p_y) \sim h / \delta \) and the absence of transitions (also with exponential accuracy) from trajectory 1 to trajectory 2 is guaranteed by the condition

\[ |p_y| \gg h / \delta \]  

(1.6)
The fulfilment of condition (1.6) is equivalent to the "usual" criterion for the quasiclassical nature of the motion (see, for example, \(^{(13)}\)) and corresponds to a "division" of the motion along upper and lower orbits\(^{(3)}\) (see Fig. 1); it is certainly satisfied for \(p_\parallel \sim p_\perp\). Precisely this simplification has been used in a large number of articles (see, for example, \(^{(13)}\)) and precisely this criterion, in essence, is cited in \(^{(41)}\). For the validity of the "true" quasiclassical treatment, however, it is also necessary to exclude quantum transitions from orbit 1 to 1' (or from 2 to 2'). The appropriate condition obviously has the form

\[(\Delta p_\parallel)^2 = (p_\perp - p_\perp) \gg \hbar/\delta.\]  

(1.7)

The fulfilment of condition (1.7), as already mentioned, essentially depends on the values of \(P_X\), \(P_Z\), and \(\epsilon\); thus, for example, for \(p_\parallel = p_\perp\), \(\epsilon = \Delta\), and \(p = \Delta\) from Eq. (1.4) it follows that

\[\rho(0) \sim (m \Delta \rho) \sim \sqrt{\gamma m \Delta} \sim \sqrt{\gamma} \sqrt{\Delta} \sim \sqrt{\Delta} \frac{h}{\Delta} \sim h / \delta,\]  

(1.8)

(see Fig. 2, where in order to be specific \(\epsilon\) is taken to be less than \(\Delta\).

A detailed analysis of the situations when the quasiclassical treatment is valid is carried out in \(^{(41)}\). Of course, the question of whether to solve this or a different physical problem quasiclassically requires a determination of what values of \(P_X\), \(P_Z\), and \(\epsilon\) are important in the problem, and whether they satisfy inequalities (1.6) and (1.7).

Now let us go on to the description of the quantum picture of the energy spectrum in a magnetic field, which is essential for what follows. In the absence of a magnetic field, for a given value of \(P_X\) the allowed energy values \(\epsilon(p_X) = \epsilon(p_X)\) are indicated by the shaded region shown on Fig. 3 (as long as \(P_X < p_\parallel\) values of \(P_Y\) and \(P_Z\) can always be found such that \(\epsilon_{\text{min}} = \Delta\); for \(P_X > p_\parallel\) one has

\[\varepsilon_{\text{min}} = \left[(P_Y^2 - P_Z^2)/2m\right]^{1/2} - \Delta^2.\]  

(1.9)

If \(H < H_1\) then one discrete level (curve 1 on Fig. 3) is split off from the continuous spectrum (\(\epsilon > 0\), \(P_X > 0\)) for each value of \(P_Y\) where the distance of this level from the "old" cutoff of the spectrum has a maximum value for \(P_X = p_\parallel\) and its order of magnitude is given by

\[\Delta - \epsilon \sim \frac{\Delta}{H_1}; \quad H_1 \sim \frac{\rho_0^2 a_0^2}{\delta} \sim 10^{-4} - 10^{-5}\text{Oe}, \quad H_1 \sim \frac{\Delta}{\Delta} \sim 10\text{Oe}.\]  

(1.10)

The \(n\)-th discrete level (for a fixed value of \(P_X\)) is split off from the continuous spectrum in a field \(H > H_1\). For \(H \gg H_1\) the picture depicted in Fig. 4 is obtained; the minimum distance of the \(n\)-th level from the gap is of the order of

\[\Delta - \epsilon_n \sim \Delta(H/H_1)^n / n!\]  

(1.11)

(measurement of the levels is conducted from below upwards). In the London case \(H_2 \sim H_{\text{cm}}\), and in the Pippard case \(H_2 \sim H_{\text{cm}}^{1/3} \ll H_1\) with \(\kappa = \delta_0 / \rho_0\).

For \(H > H_2\) the levels cross the Fermi boundary \(\epsilon = 0\); new levels appear in the magnetic field \(H = H_2\), where \(H_2 \sim H_2^2\) and \(H_2 \sim H_{\text{cm}}^2 / \Delta_L\). Their general form

\[\Delta - \epsilon \sim \Delta(H/H_2)^n / n!\]  

is shown in Fig. 5 (for convenience both \(\epsilon > 0\) and \(\epsilon < 0\) are shown in Figs. 4 and 5); the scaling in the figures is not maintained. The distance between the levels is of the order of \(\Delta(\Delta/H_2)^{1/2}\) and is appreciably larger than, for example, the quantity \(\hbar \Omega = \hbar H_2 / m c\). This is quite natural: The question is about levels resulting from "sliding" orbits with very small periods and large frequencies, which are essentially imbedded in a layer of thickness \(\delta\) (Fig. 6).

Quasiclassical quantization is determined by the usual Bohr formula \(S = \oint p_\parallel dy = 2\pi n\hbar\) (see \(^{(41,21)}\)). The more exact notation for the quantization condition has the form

\[S = 2\pi\hbar(n + \gamma),\]  

(1.12)

where the phase \(\gamma\) depends on the nature of the trajectory. Its form in the general case is derived in the Appendix (where, in particular, a mistake which was made during the calculation of one of the values of \(\gamma\) in \(^{(41)}\) is corrected). It is found that one can formulate the following simple rule for the determination of \(\gamma\); if the number of turning points is even then \(\gamma = \frac{1}{2}\), or what amounts to the same thing, \(\gamma = \frac{1}{2}\); if the number of turning points is odd, then \(\gamma = -\frac{1}{2}\).

2. PHYSICAL PICTURE OF THE QUANTIZED OSCILLATIONS

The periodic splitting-off of the discrete levels from the continuous band spectrum would appear as if it might lead to quantized oscillations of the different physical quantities. Let us consider, however, the situation which arises in more detail. At low temperatures the levels which are split-off from above \(\epsilon \gg \Delta\), see Fig. 4) are occupied to an exponentially small extent (to the extent \(\exp \{-\Delta / T\}\); at high temperatures \(T \sim \Delta\) the thermal "broadening" of a given level (i.e., the possibility of

\[\text{\textsuperscript{31}}\text{This rule is valid both for the excitations in a superconductor as well as for "ordinary" particles and quasiparticles.}\]
transitions to other levels and from other levels) leads to an exponentially small amplitude of the oscillations (small with respect to $T/(\delta\epsilon)$, where $\delta\epsilon$ is the distance between the discrete levels). The levels for transitions to other levels and from other levels) leads to oscillations. At low temperatures the total number of states $N$ for $\epsilon < 0$ is obviously conserved with exponential accuracy (in $\Delta/|\epsilon|$) and since all thermodynamic quantities can be expressed in terms of the number of states since

$$\Omega(\mu) = -\iint N(\mu') d\mu'',$$

the amplitude of the corresponding oscillations again turns out to be exponentially small. Therefore the observable oscillations appear when the discrete levels cross the Fermi boundary $\epsilon = 0$, i.e., for $H > H_s$; thereby the gap in the excitation spectrum vanishes and the oscillating picture develops in principle just like in a normal metal. In normal metals the oscillations are a bulk property; in the interior of a metal the discrete spectrum only depends on a single continuous parameter ($P_2$) while degeneracy exists with respect to the position of the center of the orbit, which is determined by $P_3$. In superconductors the spectrum is discrete only near the surface, where the quantizing magnetic field is not small, and it depends on both parameters ($P_X, P_2$). Therefore the amplitude of the oscillations contains the first power of the parameter $(\delta\epsilon)/\epsilon \sim \rho/\epsilon \eta_{\text{max}}$, which determines the fraction of the participating energies. (In a normal metal the corresponding parameter $h_0/\epsilon$ enters raised to the power $1/2$.)

Since with an increase of the magnetic field the superconductor changes into a normal metal, specifically the "superconducting" oscillations, incipient for $H > H_s$, continue only up to $H_s \sim \kappa H_s \gg H_s$ in the London case and up to $H_s \sim \kappa^{1/2} H_s \gg H_s$ in the Pippard case. It is only necessary to mention that, beginning with $H_s \sim \kappa^{-1} H_s \ll H_s$, in the case of a London superconductor the picture is complicated because of the appearance of vortex lines, which leads to a two-dimensional problem. However, since the distance between the vortices in fields $H \gg H_s$ is much smaller than the size of the orbit $(r_0)^{1/2}$ ($r_0$ is the Larmor radius), then in the $x$ direction the dependence of $A$ on $x$ is averaged and in the basic approximation one can restrict oneself to the one-dimensional case (with $\Delta$, of course, depending on $y$). All of the stated information can be directly transferred to the case of an arbitrary dispersion law and arbitrary dependence of $\Delta(\epsilon)$ in Eqs. (1.1) and (1.2). It is clear that the specific form of the oscillations depends on $(\epsilon(\mathbf{p})$ and on $\Delta(\mathbf{p})$; it is important, however, that the basic formula (1.12) of quasiclassical quantization is retained. Therefore, the period of the oscillations will always be determined by that change of the field $\Delta H$ for which the number of levels under the Fermi boundary changes by one:

$$\Delta N_{\text{surf}}(\epsilon = 0, H) = 2\pi h.$$. (2.1)

The extremum of $S(\mathbf{P}_X, \mathbf{P}_2)$ is taken with respect to $\mathbf{P}_X$ and $\mathbf{P}_2$, which guarantees the maximum density of oscillation of the determined states. It is curious that the extremal nature of the number of levels as a function of $P_2$ and $P_3$ indicates periodicity of the motion of the excitations along the appropriate orbits in coordinate space. In fact, the displacement during the period $T$ is given by

$$\Delta \mathbf{r} = \frac{\partial S_{\text{surf}}}{\partial \mathbf{P}_2} \frac{\partial \mathbf{P}_2}{\partial \mathbf{P}_3} = -\frac{\partial S}{\partial \mathbf{P}_3}, \quad \Delta \mathbf{r} = -\frac{\partial S}{\partial \mathbf{P}_3} \Delta \mathbf{P}_3$$

and for $S = S_{\text{surf}}$ we have $(\Delta \mathbf{r})_T = (\Delta \mathbf{r})_T = 0$. The shape of these orbits is shown in Fig. 7. We note that analogous cross sections are also distinguished in cyclotron resonance, in the de Haas–van Alphen effect, etc.

![FIG 7](image)

As usual, the temperature dependence of the amplitude of the oscillations is determined by the distance between the levels:

$$\Delta \mathbf{r} = h_0 \frac{\partial S}{\partial \mathbf{P}_2} = 2\pi h / \frac{\partial S_{\text{surf}}}{\partial \mathbf{P}_2} \Delta \mathbf{P}_3.$$. (2.3)

Experimental investigation of the oscillations makes it possible, in principle, to determine two functions of $H$ (namely, $\delta S/\delta \epsilon$ and $\delta S/\delta H$), where in contrast to the case of normal metals the dependence on the direction and magnitude of the field is not a universal function but depends on the specific form of $\xi(\mathbf{p})$ and $\Delta(\mathbf{p})$. Thus, experimentally one can obtain two functions of three variables whereas all of the quantities $(\xi(\mathbf{p}), \Delta(\mathbf{p}),$ and $\delta S/\partial \mathbf{p})$ entering into formulas (2.1) and (2.2) are evaluated only on the Fermi surface and therefore only depend on two variables. This implies the possibility in principle of experimentally establishing all of the functions which enter into the theory.\(^4\)

In conclusion, before going on immediately to the calculations, let us turn our attention to the nature of the conditions on Maxwell's equations at the interface between the metal and vacuum. Since the following discussions pertain to both superconductors and to normal metals, we shall demonstrate them for the example of normal metals. As is well known, the boundary conditions on Maxwell's equations are considered to be continuity of the tangential components, $E_t$ and $H_t$, of the fields, and also continuity of the normal components $B_n$ and $D_n$. These conditions are obtained after a suitable averaging of the microscopic equations of the field. The problem of specific interest to us about the quantum oscillations is the presence of several "microscopic" distances of different orders of magnitude, over which the macroscopic quantities are formed. These distances are: the intercharge distance $a \sim h_0 P_3$, the Larmor radius $r_0$, and the mean free path length $l$. Upon averaging over distances $\Lambda \gg a, r_0, l$ the usual macroscopic

\(^4\)The point is that knowledge of a function of three variables is equivalent to knowledge of an infinite number of functions of two variables, for example:

$$f(x, y, z) = \sum_{k=0}^{\infty} \frac{\partial^k f(x, y)}{k!} h_0 \frac{\partial y}{\partial \mathbf{P}_3} \delta^k \Delta \mathbf{P}_3.$$
equations and boundary conditions are naturally obtained.

However, as a rule we are interested in averaging over distances $a \ll \Lambda \ll r, l$. In this connection in the equation connected with the magnetic field,

$$\text{rot} \mathbf{B} = \frac{4\pi}{c} j,$$  
(2.4)

both terms which are determined by distances of the order of a (the spin paramagnetism $M_0$, the constant Landau diamagnetism $M_L$, and the lattice magnetism $M_L$) as well as the oscillating diamagnetic moment, which changes over distances of the order of r, enter into $\Lambda$. Therefore, in connection with the well-known derivation of the boundary conditions, the integral of the oscillating term on the right hand side of Eq. (2.4) turns out to be small (to the extent that $\Lambda/r$ is small) and the quantity $(\mathbf{B} - 4\pi M_0 - 4\pi M_L - 4\pi M_L) r$ turns out to be continuous. Since we are not interested in small monotonic terms, in what follows we shall write the boundary condition simply as the continuity of the vector $\mathbf{B}$.

Let us describe the plan of the calculation of the quantum oscillations to be given below. As already mentioned the oscillations are always associated with quasiclassical states; therefore Sec. 3 is devoted to a description of the quasiclassical wave functions and Green's functions. The necessity for a description of both the wave functions and the Green's functions is connected with the following. In order to calculate the energy gap and the current density it is necessary to carry out a summation over all the states. In weak fields ($H < H_s \ll H_g$) in the presence of an essential "non-quasiclassical nature" in the energy spectrum— that is, a gap whose magnitude is appreciably larger than the distance between the levels—it is more convenient to use the Green's functions since the quasiclassical Poisson summation in this case turns out to be extremely complicated. Namely, in Sec. 4 the exponential smallness of the amplitude of the quantum oscillations is proved for $H < H_s$ with the aid of the Green's functions. In strong fields ($H > H_s$) the gap in the spectrum of the states of interest to us is absent, and the calculations turn out to be simpler in terms of the wave functions (Sec. 4). Finally, Sec. 5 is devoted to the determination of the oscillations of an easily measured physical quantity—the surface impedance.

3. QUASICLASSICAL WAVE FUNCTIONS AND GREEN'S FUNCTIONS

The form of the wave functions of the excitations in a superconductor in the quasiclassical region was derived in [6]. In [4] the asymptotic form of the wave functions is written down for two limiting cases in the region sufficiently close to the turning points, and the value of the phase $\gamma$ in the Bohr quantization rules is found. Let us carry out an investigation of the general case in a symmetric notation. Strictly discrete levels correspond to finite phase trajectories. For $p_0 \sim pF$ the corresponding trajectories, as follows from Eq. (1.4), have the form shown in Fig. 8a (the dotted line indicates the motion which would formally occur in the absence of the boundary at $y = 0$); with the reduction of $(pF - p_0)$ the trajectory takes the form shown in Fig. 8b; finally, for $(pF - p_0) \lesssim 0$ the trajectory is shown on Fig. 8c.

Now let us find the wave functions in the quasiclassical regions, i.e., sufficiently far away from the turning points $y_0, y_1$. The general equation for the wave functions $\Psi$ is given by

$$
\begin{pmatrix}
\xi - \bar{\rho} \\
-\bar{\rho} - \xi
\end{pmatrix}
\begin{pmatrix}
\psi \\
\psi_0
\end{pmatrix}
= \begin{pmatrix}
0 \\
1
\end{pmatrix},
(3.1)
$$

In (3.1) the small terms $(eA v_X/mc)^2$ are at once discarded. We shall seek the solution in the form

$$\Psi = (F_0 \cdot hF_0) \exp \left\{ \frac{i}{\hbar} \int \left( p_0 + h p_d \right) dy + \frac{i p_x}{\hbar} + \frac{i p_z}{\hbar} \right\}. 
(3.2)
$$

Substitution of (3.2) into (3.1) leads, in the zero-order approximation in $\hbar$, to the following homogeneous equation for $F_0$:

$$
\left( \hat{H}_0 - \epsilon \right) F_0 = \begin{pmatrix}
\frac{p_0^2}{2m} - \frac{p_0^2 - p_e^2}{2m} - \rho - \epsilon - \Delta \\
- \Delta_1 \frac{p_0^2}{2m} + \frac{p_0^2 - p_e^2}{2m} - \rho - \epsilon
\end{pmatrix} F_0 = 0, 
(3.3)
$$

which contains $p_0$ as a parameter. The condition for the solvability of the homogeneous equation (3.3) determines $p_0$ coinciding with the "classical" $p_F(y)$ in Eq. (1.4); and the solution of (3.3) associated with this $p_F$ gives the "classical" solutions for $u$ and $v$. In the next approximation in $\hbar$ an inhomogeneous equation is obtained for $F_0$:

$$
\left( \hat{H}_0 - \epsilon \right) F_0 = \frac{p_0}{m_0} \cdot \begin{pmatrix}
\frac{d F_0}{dy} + \frac{1}{2} F_0 \frac{d}{dy} (\ln p_0) - p_0 F_0 \\
0
\end{pmatrix},
\begin{pmatrix}
\gamma \\
0
\end{pmatrix},
(3.4)
$$

Since the homogeneous equation (3.3) corresponding to (3.4) has a nontrivial solution $F_0$, for (3.4) to have a solution the right-hand side (which depends on $p_0$ as a parameter) must be orthogonal to the solution of the homogeneous equation which is the adjoint of (3.3), that is, to $F_0$:

$$
\left\{ \frac{d F_0}{dy} + \frac{1}{2} F_0 \frac{d}{dy} (\ln p_0) \right\} \delta = p_0 F_0. 
(3.4a)
$$
From Eq. (3.4a) it follows that
\[ p_i = \frac{i}{\hbar} \frac{d}{dy} \ln \{ \rho_i(F_0 F_i) \}. \] (3.5)

Substituting (3.5) into (3.2), in the basic approximation (in order to write down this approximation, knowledge of \( F_0 \) is not required) we obtain the natural quasiclassical formula
\[ \Psi = C \sum_{\alpha} \exp \left\{ \frac{i}{2\hbar} \int \rho_\alpha dy \right\}, \]
\[ F_i = \left\{ \begin{array}{ll} \sqrt{v_i} \int \rho_i dy \right\}, \] (3.6)

In connection with the fact that the phase trajectory has several branches, finally one can write down the following answer for \( \psi \):
\[ \Psi = C \sum_{\alpha} \exp \left\{ \frac{i}{2\hbar} \int \rho_\alpha dy \right\}. \] (3.7)

For \( y < y_1 \), the summation in formula (3.7) goes over all values \( \rho_\alpha(y) \) which are classically possible, and for \( y > y_1 \) it goes over those \( \rho_\alpha(y) \) for which \( \Im \rho_\alpha(y) > 0 \), which guarantees attenuation of the wave function at infinity. In order to determine the phase factors \( \exp \{ i \rho_\alpha \} \) it is necessary, as usual, to match the quasiclassical solutions to the right and to the left of the turning points \( y_0 \) and \( y_1 \), near which the quasiclassical solution is not admissible, and it is also necessary to take the boundary condition \( \Psi |_{y=0} = 0 \) at the surface of the metal into account. The constant \( C \) is found from the normalization conditions:
\[ C^2 \sum_{\alpha} \int \rho_\alpha dy = C^2 \int \frac{d\rho_\alpha}{\sqrt{\rho_\alpha}} = C^2 \int \frac{\rho_\alpha^2 dy}{\sqrt{\rho_\alpha}}, \] (3.8)

It is obvious that all of the integrals over \( y \) appearing in Eq. (3.8) are small for \( \alpha = \beta \) (to the extent that the quasiclassical parameter \( 1/n \) is small, where \( n \) denotes the number of the level); with the same accuracy one can assume that the integration in (3.8) only goes over the region \( 0 \leq y \leq y_\alpha \). As a result for \( C \) and \( \Psi_y \) we obtain
\[ \Psi = \frac{1}{\sqrt{\delta \delta \sqrt{\delta \delta}},} \]
\[ \Psi = \frac{1}{\sqrt{\delta \delta \sqrt{\delta \delta}},} \]
\[ \Psi = \frac{1}{\sqrt{\delta \delta \sqrt{\delta \delta}},} \]
\[ \Psi = \frac{1}{\sqrt{\delta \delta \sqrt{\delta \delta}},} \]
\[ \Psi = \frac{1}{\sqrt{\delta \delta \sqrt{\delta \delta}},} \]

From the last discussion it is also seen that to the accuracy we are interested in the phase factors \( \exp \{ i \rho_\alpha \} \) only affect the phase \( \gamma \) in the conditions for quasiclassical quantization (see Appendix 1).

Now let us find the quasiclassical temperature-dependent Green's function
\[ G(w, F_0 F_1, y, y') = \left\{ \begin{array}{ll} G_1(w) \end{array} \right\}. \] (3.11)

Here \( \omega_0 = (2s + 1) \pi T \) (\( s \) is a natural number). For this purpose let us utilize the following property of the Gor'kov equations, namely: \( G(\omega_0) = g_1(\omega_0) \), where \( g_1(\omega) \) is the solution of the corresponding equations for the time-dependent Green's functions obtained under the condition that \( \omega \neq \epsilon_n (F_0, F_1) \), where \( \epsilon_n \) denote the energy levels of the excitations in the superconductor. The equation for the determination of \( g_1(\omega) \) has the same form as (3.1); it is only necessary to replace \( \epsilon \) by \( \omega_0 \) and for \( y = 0 \) and \( y = y' \) it is necessary to establish the following boundary conditions: \( g_1(0, y') = 0 \), \( g_1(y'-0, y') = g_1(y'+0, y') \).
these quantities it is convenient to express them in terms of the wave functions of the excitations.

The corresponding formulas have the form

\[ \Delta = |\lambda| \sum_{\text{states}} n_v^{c, n_v(r_c)}, \]

\[ j_i = j = \sum_{\text{states}} \left( \frac{e}{m} \right) \left( \frac{e}{c} A_i \right) n_v^{c, n_v(r_c)}. \]  

(4.1)

In each of the relations (4.1) there is an integration over the continuum and a summation over the discrete spectrum. As already discussed, the oscillating term of interest to us is contained only in the summations over the discrete states. For their computation we shall use Poisson's formula,\(^7\) and the wave functions of the excitations and the energy spectrum we take in the quasiclassical approximation. As a result, for the oscillating correction to the current density we have

\[ j_i = \frac{e}{2m} \text{Re} \sum_{\nu_{m}} \int dP_x dP_z \left( \frac{e}{m} \right) \left( \frac{e}{c} A_i \right) u_{\nu_{m}}^{c, n_v(r_c)} \exp \{ i \alpha \}. \]

(4.2)

\( N_{\text{min}} \) and \( N_{\text{max}} \) are the minimum and maximum values for the number of levels associated with fixed values of \( P_x \) and \( P_z \). Now in the integral over \( n \) let us change the variable according to the formula \( n = \theta(k) \), after which one can easily perform the integration over the energy under the following assumptions: \( |\epsilon_{\text{min, max}}| \gg T_s \), \( \text{sign} \epsilon_{\text{min}} = - \text{sign} \epsilon_{\text{max}} \),\(^7\) where \( \epsilon_{\text{min}} \) and \( \epsilon_{\text{max}} \) denote the upper and lower boundaries of the discrete spectrum for fixed values of \( P_x \) and \( P_z \). As a result we find

\[ j_i = \text{Re} \sum_{\nu_{m}} \int dP_x dP_z \left( \frac{e}{m} \right) \left( \frac{e}{c} A_i \right) u_{\nu_{m}}^{c, n_v(r_c)} \exp \{ i \alpha \}. \]

(4.3)

The integration over \( P_x \) and \( P_z \), which appears in formula (4.3), is elementary to perform using the method of steepest descents, after which we obtain the following answer:

\[ \Delta = \frac{1}{2m} \sum_{\nu_{m}} \left( \frac{e}{m} \right) \left( \frac{e}{c} A_i \right) u_{\nu_{m}}^{c, n_v(r_c)} \exp \{ i \alpha \}. \]

(4.4)

where \( p_x^{\text{extr}} \) and \( p_z^{\text{extr}} \) are determined from the equations \( \partial \Delta / \partial P_x = 0 \) and \( \partial \Delta / \partial P_z = 0 \); \( \varphi = 0 \) if \( \partial \Delta / \partial P_x \) and \( \partial \Delta / \partial P_z \) have the same sign, and \( \varphi = \pi / 2 \) if they have different signs. Similar calculations for the gap \( \Delta_1 \) lead to the following answer:

\[ \Delta_1 = \frac{1}{2m} \sum_{\nu_{m}} \left( \frac{e}{m} \right) \left( \frac{e}{c} A_i \right) u_{\nu_{m}}^{c, n_v(r_c)} \exp \{ i \alpha \}. \]

(4.5)

As already referred to in (2.2), the extremal cross sections correspond to excitations with average velocity equal to zero. The absence of a gap in the spectrum was very essential in all of the calculations which have been carried out. In the presence of a gap (i.e., in a magnetic field \( H < H_2 \)) it is more convenient to write the energy gap and the current density in terms of the Green's function:\(^{11}\)

\[ j_i = \frac{T}{2\pi h^2} \int dP_x dP_z \sum_{\nu} \left( \frac{e}{c} A_i \right) G(\nu), \]

(4.6)

\[ \Delta = |\lambda| \sum_{\nu} dP_x dP_z \frac{T}{4\pi e^2 h^2} F^*(\nu). \]

From formulas (4.6) and (3.13) it follows that for \( \rho < \Delta \) the series in (4.6) are represented by a sum of exponentially small terms, i.e., the series are exponentially small. In fact, as is clear from (3.17), if \( \rho < \Delta \), then for \( T = 0 \) one finds \( \text{Im} S \neq 0 \) and \( G \sim \exp \{- |\text{Im} S| \} \).

Now let us demonstrate this same result for the exactly solvable model of Pincus. With the aid of (3.15) \( j_X \) and \( \Delta \) are easily calculated according to formulas (4.6). One can represent \( G_m \) as the sum of two terms: \( G_m^{\text{pp}} \)-the smooth part and \( G_m^{\text{osc}} \)-the oscillating part. One can verify that the current \( j_X \) calculated using \( G_m^{\text{osc}} \), is equal to the expression cited in (31) if the vector potential is taken as (3.14) plus a small smooth correction, connected with the effect of the surface. The function \( G_m^{\text{pp}} \) is proportional to \( \exp \{- \beta/|\alpha_m|^2 \} \), and for \( |\rho| < \Delta \) (see (3.16)) \( G_m^{\text{osc}} \sim \exp \{- \beta/|\alpha_m|^2 \} \) — which is an exponentially small quantity. Thus, the oscillations exist in that region of magnetic field where \( |\rho| > \Delta \). Similarly, for the temperature dependence of the amplitude of the oscillations \( (|\rho| > \Delta) \) we obtain

\[ G_m^{\text{osc}} \sim \exp \left\{ - \frac{2\pi T}{(h/2m)^2} \right\}. \]

5. QUANTUM OSCILLATIONS OF THE SURFACE IMPEDANCE AND OF THE GAP

As is well known, the surface impedance \( Z(\omega) \) is given by

\[ Z(\omega) = \frac{4\pi \alpha_0 A_s(0)}{c} \frac{\bar{B}_s}{B_s}, \]

(5.1)

Here \( A_s(0) \) and \( B_s(0) \) are the variable components, respectively, of the vector potential and of the magnetic induction on the boundary of the superconductor. For not too large frequencies, the \( \{ \mathbf{B} \} \) in Maxwell's equations for the determination of the field can be taken in the form \( \{ \mathbf{B} \} = \tilde{Q} \mathbf{A} + \{ \mathbf{B} \} \), where \( \tilde{Q} \) is the operator connecting \( \mathbf{A} \) and \( \mathbf{A} \) in the "usual" case when no oscillations are present,\(^{11}\) and \( \mathbf{B} \) is calculated according to formula (4.5). Let \( B_s(0) \) be smaller than the period \( \Delta H \) of the oscillations and, of course, smaller than the external constant magnetic field \( H \). In this case the equation

\[ \mathbf{rot} \mathbf{B} + \mathbf{rot} \mathbf{B} = \frac{4\pi}{c} \left( \tilde{Q} \mathbf{A} + \tilde{Q} \mathbf{A} \right) \]

can be easily solved in the linear approximation with respect to \( B_s \). From (5.2) we have the following equations for the determination of \( \mathbf{B} \) and \( \mathbf{B}^\ast \):

\[ \mathbf{rot} \mathbf{B} = \frac{4\pi}{c} \tilde{Q} \mathbf{A} + \frac{4\pi}{c} \{ \mathbf{B} \}, \]

(5.3)

\[ \mathbf{rot} \mathbf{B}^\ast = \frac{4\pi}{c} \tilde{Q} \mathbf{A} + \frac{4\pi}{c} \{ \mathbf{B} \} \]

(5.4)
In the last term of Eq. (5.4) it is necessary to separate out the part which is linear in \( B_\omega \). It is clear that in this connection it is necessary to keep the dependence on \( B_\omega \) only in the argument of the cosine in (4.5). Let us consider the expression for \( n_{\text{extr}} \) in more detail:

\[
2\pi n_{\text{extr}} = \int_{0}^{\infty} B_\omega \frac{d\omega}{2\pi}.
\]

For the determination of \( n_{\text{extr}} \) given by solution of Eq. (5.3)) is given by

\[
\rho(y) + \rho_\omega(y) = \int_{\omega_0}^{\omega} \frac{d\omega}{2\pi} \rho_\omega \left( B - B_\omega \right) \frac{d\omega}{2\pi}.
\]

Having made use of formula (5.5) for the correction to the current density which is linear in \( B_\omega \) and of interest to us, we obtain

\[
\langle \beta \rangle = -\sum_{n} \frac{eP_{\text{extr}}^{n}}{m} \left| \Psi \right|_{n}^{2} \theta(k_{n}^{0}) \times \left( \frac{\partial_{n}}{\partial P_{B}^{n}} \right)_{\text{extr}}^{-1} \frac{\alpha}{\alpha_{B}} d\gamma \sin(2\pi k_{n} \gamma, \phi).
\]  

We note that in formula (5.7) for \( \rho_\omega \) and \( B_\omega \) one can use the basic approximation, that is, one can use the solution of the equations

\[
\text{rot}B = \frac{4\pi_{B}}{c} QA, \quad \text{rot}B_{\omega} = \frac{4\pi_{B}}{c} QB_{\omega}.
\]

From (5.8) it is clear that \( \rho_\omega \) and \( \rho \) decrease inside the metal in the same way and differ only in their amplitudes, that is,

\[
\rho_{\omega} = \frac{\rho_{\omega}(0)}{B_{\omega}(0)} \rho, \quad \text{and} \quad \langle \beta \rangle = B_{\omega}(0) \frac{\partial f}{\partial H}.
\]

In Eq. (5.4) we change to Fourier components, having continued the field \( B_\omega \) onto the semi-axis \( y < 0 \) in an even manner. We recall that this corresponds to making the assumption about the spectral nature of the reflection of the excitations at the boundary of the metal. After this the equation is easily solved, and one obtains the following answer for the surface impedance:

\[
Z = \frac{2\pi}{c} \int_{0}^{\infty} \left[ 1 + \frac{4\pi^{2}}{c} \left( \frac{\partial f}{\partial H} \right) \right] (q^{2} + Q(q))^{-1} dq.
\]

We note that the attenuation depth of a static field (the solution of Eq. (5.3)) is given by

\[
\delta_{n} = \frac{2}{\alpha} \int_{0}^{\infty} \left[ 1 + \frac{4\pi^{2}}{Bc^{2}} \right] (q^{2} + Q(q))^{-1} dq.
\]

Now let us again return to the quantum oscillations of the gap \( \Delta \), which it is natural to represent in the following form:

\[
\Delta = \Delta(y) + \Delta(y).
\]

The Green's function \( F'_{\omega}(\Delta(y)) \) in turn is the sum of the contribution \( F'_{\omega}(\Delta(y)) \) from the continuous part of the spectrum and the contribution \( F'_{\omega}(\Delta(y)) \) from the discrete part of the spectrum. During the calculation of \( F'_{\omega}(\Delta(y)) \) one can neglect the small correction \( \Delta(y) \); it is precisely this quantity \( \Delta = \frac{\Delta}{\Delta} \) which was calculated in Sec. 4. As a result, the following equation is obtained for the determination of the gap:

\[
\Delta(y) + \Delta(y) = F'_{\omega}(\Delta(y) + \Delta(y)),
\]

The solution of (5.14) in a magnetic field of arbitrary magnitude is an extremely complicated problem. However, if it is assumed that \( H_{0} < H < H_{c} \), then it is found that in the basic approximation one can use the "usual" expression for \( F'_{\omega}(\Delta) \) after this it is not difficult to obtain the result that \( \Delta = \frac{\Delta}{\Delta} \ln (\Theta_{D}/\Delta_{0}) \), having substituted here the value of \( \Delta_{0} \) from (5.4), we finally obtain

\[
\Delta(y) = \sum_{n} \frac{u_{n}}{2m_{B}^{2}} \frac{4\pi_{B}}{c} e \left( \omega_{n} \right) \cos(2\pi k_{n} \gamma, \phi).
\]  

6. DISCUSSION OF THE RESULTS

First of all let us consider the temperature dependence of the amplitude of the oscillations. As is clear from formula (5.10), the temperature dependence is determined by the function \( \theta(k_{n} \gamma, \phi) \). If the temperature \( T > (6c_{\text{extr}}/2\pi)^{1/3} \), then the amplitude of the oscillations is proportional to \( 4\pi T / \theta(k_{n} \gamma, \phi) \exp \left[ -2\pi T / (6\theta_{\text{extr}}) \right] \), i.e., it is exponentially small. For low temperatures \( T < (6\theta_{\text{extr}}/2\pi)^{1/3} \), \( \theta(k_{n} \gamma, \phi) = 1 \).

Now let us estimate the relative amplitude, for example, of the oscillations of the surface impedance. It is most convenient to do this by assuming that the dispersion law is quadratic, and the field decreases deep inside the metal according to the exponential law \( \exp \left[ -y/\theta_{0} \right] \). Then \( \rho_{\omega} = \rho_{0} > \Delta \). With the aid of these formulas we obtain the following result for the relative amplitude of the oscillations of the surface amplitude, \( \Delta Z / Z \):

\[
\frac{\Delta Z}{Z} \sim \frac{1}{\delta_{n}} \cos \left( \frac{\theta_{n}}{\theta_{0}} \right) \omega \left( \frac{2\pi n^{2}}{\Theta_{D}^{2}} \right)^{1/3} \theta \left( \frac{2\pi n}{\Theta_{D}} \right),
\]

where \( \sim \) denotes the corresponding oscillating part. For the calculation of both \( \Delta_{0} \) and \( \Delta \) one must use the self-consistency condition:

\[
\Delta = |\Delta F'_{\omega} (\Delta, y) |.
\]

The Green's function \( F'_{\omega} (\Delta, y) \) in turn is the sum of the contribution \( F'_{\omega} (\Delta, y) \) from the continuous part of the spectrum and the contribution \( F'_{\omega} (\Delta, y) \) from the discrete part of the spectrum. During the calculation of \( F'_{\omega} (\Delta, y) \) one can neglect the small correction \( \Delta (y) \); it is precisely this quantity \( \Delta = \frac{\Delta}{\Delta} \) which was calculated in Sec. 4. As a result, the following equation is obtained for the determination of the gap:

\[
\Delta (y) + \Delta (y) = F'_{\omega} (\Delta (y) + \Delta (y) + \Delta (y)),
\]

The solution of (5.14) in a magnetic field of arbitrary magnitude is an extremely complicated problem. However, if it is assumed that \( H_{0} < H < H_{c} \), then it is found that in the basic approximation one can use the "usual" expression for \( F'_{\omega} (\Delta) \) after this it is not difficult to obtain the result that \( \Delta = \frac{\Delta}{\Delta} \ln (\Theta_{D}/\Delta_{0}) \), having substituted here the value of \( \Delta_{0} \) from (5.4), we finally obtain

\[
\Delta (y) = \sum_{n} \frac{u_{n}}{2m_{B}^{2}} \frac{4\pi_{B}}{c} e \left( \omega_{n} \right) \cos(2\pi k_{n} \gamma, \phi). \]  

The similar local relation is, of course, an approximation since strictly speaking \( \Delta = \frac{\Delta}{\Delta} (\Delta_{s}) \). We further note that the smooth (in the sense of the dependence on given values of the gap and of the magnetic field) part of the current density also contains oscillations due to \( \Delta \); however, this part is \((\Theta_{D} / \Theta_{0} k_{B} \theta_{0})^{1/3} \) times smaller than \( \psi \).
magnetic field. At low temperatures (T < Tc) the smooth part of the surface impedance depends on the magnetic field weakly due to the fact that the numerical coefficient of (H/Z)(dZ/dH) ~ 10^{-12} to 10^{-4} for the oscillating correction the corresponding quantity is of the order of

$$\frac{H_s d\Delta Z}{Z dH_s} \sim 10^{-12} n_{exr} (\frac{\lambda_c}{h})^6 \sim 10^{-4} - 10^{-1}. \quad (6.2)$$

From here it is seen that the quantum oscillations of the derivative of the surface impedance with respect to the magnetic field may turn out to be dominant. If the dispersion law is cylindrical, then all of the relative amplitudes of the oscillations obviously turn out to be (n_{exr})^{1/2} times larger.

Now let us return to the question of the possibility of establishing the dependences of the amplitude of the oscillations through the functions (2n/\Delta H) and (2n/\Delta c) (in what follows the subscript extr is omitted everywhere).

Let us assume, for simplicity, that the dispersion law is known to us, and the field falls off inside the metal according to an exponential law. Then

$$\frac{\partial \alpha}{\partial H_s} = \frac{\delta}{2nH_s} \frac{d}{dH_s} (p^{(1)}(0) - p^{(0)}(0)), \quad \frac{\partial \alpha}{\partial H_s} = \frac{\delta}{2nH_s} (p^0(0) - p^0(0)), \quad (6.3)$$

where p^{(1)}(0) and p^{(0)}(0) are possible values of the momentum p_y of the excitations on the boundary of the superconductor. From the picture under consideration of the classical motion of the excitations, it is clear that the surface states are connected with those excitations whose velocity along the y axis is small. Making use of this property, it is not difficult to obtain the result that

$$p^{(1)}(0) - p^{(0)}(0) = 2 \pi 2m^* (\mu_x + \frac{\lambda_c}{\Delta} - \lambda),$$

$$\mu_x = \frac{\lambda_c}{\Delta} (p_y^0(0), p_x, p_z), \quad m^* = \frac{\delta \alpha}{\delta p_x} (p^{(0)}(0), p_x, p_z),$$

where p^{(0)}(0) = p^{(0)}(P_x, P_z) is the solution of the equation \((\delta p_y / \delta p_y) = 0\). From relations (6.3) it now follows that \(\Delta(p) = \rho_m m^*(q_y, m^*)^4 - 1)^{1/4} / \rho_m q_y. \quad (6.4)\)

Here

$$\varphi = \pi H_0 \delta \frac{\partial \alpha}{\partial H_s} \frac{\partial \alpha}{\partial \delta}, \quad g = \pi H_0 \delta \frac{\partial \alpha}{\partial H_s} \frac{\partial \alpha}{\partial \delta}$$

are directly measurable quantities. Formula (6.4) solves the problem which has been posed, provided the extremal point \(P_x, P_z\) is known. We note that with an accuracy of the order of \((\Delta \rho (H_c))^{1/2} \sim 5 \text{ to } 20\%\), we have the following equations for the determination of the values \(P_x, P_z\) of interest to us:

$$\frac{\partial \varphi_x}{\partial P_x} = 0, \quad \frac{\partial \varphi_x}{\partial P_z} = \varphi(H_x) \frac{e}{g(H_z)} \frac{\varphi_x}{\varphi_z} = \frac{e}{g(H_z)} \varphi_x \frac{1}{2e \delta E_x} \quad (6.5)$$

7. CONCLUSIONS

In this article the quantum oscillations of the energy gap, current density, and surface impedance in superconductors have been investigated, and the corresponding expressions (5.15), (4.4), and (5.10) have been derived for these oscillations.

It has been ascertained that the oscillations in superconductors have an essential singularity with respect to the magnetic field—said singularities being present only for \(H > H_s\) that is, when the gap in the energy spectrum vanishes. It is shown (see expression (6.1)) that the relative amplitude of the oscillations of the gap and of the current density is determined by the quantity

$$\frac{4}{n} \left(\frac{\alpha}{\delta \alpha}ight)^{1/4} \exp \left(-\frac{2\Delta y}{\delta \alpha}ight),$$

and for the surface impedance the relative amplitude is determined by

$$\left(\frac{\alpha}{\delta \alpha}ight)^{1/4} \exp \left(-\frac{2\Delta y}{\delta \alpha}ight),$$

provided that the Fermi surface is spherical; the relative amplitude is \(n^{1/2}\) times larger if the Fermi surface is cylindrical; here \(n\) denotes the number of discrete levels.

It is shown (see Sec. 2) that from the temperature dependence of \(2n/\Delta c\) and from the period \(2n/\Delta c\) of the oscillations one can, in principle, establish \(\varphi_s\) and \(\Delta(p)\) as functions of \(p\). In Sec. 6, under the simplest assumptions, formula (6.4) is derived for the determination of \(\Delta(p)\) from \((2n/\Delta c)\) and \((2n/\Delta c)\). The result is obtained that the condition on Maxwell's equations at the interface between the metal and vacuum is the continuity of \(\mathbf{B}\).

A general rule is found for determining the phase \(\gamma\) in the Bohr quantization conditions (see the Appendix): \(\gamma = \frac{1}{2}\) if the number of turning points is even and \(\gamma = -\frac{1}{2}\) if the number of turning points is odd.

APPENDIX

Let the parameters \(e, P_x, P_z\) be such that the classical trajectory corresponding to the motion of the excitation has the form (see Figs. 8a and 8b) where the quasiclassical approximation is applicable in the region \(\{0, y_b\} \leq y \leq y_f\). Here \(\{a, b\} = \max(a, b)\). Let us find the value of the phase \(\gamma\) in the Bohr quantization rules for such trajectories. With quasiclassical accuracy the equation for the component \(v\) of the wave function \(\psi = \sqrt{\psi}\) has the form

$$\left\{\left(\frac{\hbar^2 p_x^2}{2m_y} + \mu_x\right) \Delta^2 (e + \rho) \right\} v = 0. \quad (A.1)$$

First of all let us write the quasiclassical solution in the interval \(\{0, y_b\} \leq y \leq y_f\). It has the form

$$v = A_1 \exp \left(-\frac{i}{\hbar} \int p, dy'\right) - A_1 \exp \left(-\frac{i}{\hbar} \int p, dy'\right)^{+},$$

$$+ B_1 \exp \left(-\frac{i}{\hbar} \int p, dy'\right) - B_1 \exp \left(-\frac{i}{\hbar} \int p, dy'\right)^{+}. \quad (A.2)$$

In order to determine the coefficients \(A_1, B_1\) we match this solution with the asymptotic exact solution of Eq. (A.1) to the left of the point \(y_f\). Near this point, to the accuracy we require Eq. (A.1) has the form

$$v^{(y_f)} + 2\nu k \frac{d^2}{dy^2} + k_0^2 + \beta (y - y_f) v = 6,$$

$$k_0^2 = 2m_y / \hbar^2,$$

$$\beta = -2\Delta p (y_f) / \Delta t \geq 0.$$
Solving this equation by Laplace's method, we obtain
\[ v = C \exp \left\{ \Delta y + \frac{1}{\beta} \left( \frac{\pi}{2} + \frac{2}{3} k_0 t^2 + k_1 t^4 \right) \right\}. \]  \hspace{1cm} (A.3)

We choose the contour \( L \) so that the solution (A.3) decreases at infinity. In order to achieve this, it is clear that it is sufficient to position the ends of the integration contour in the region where \( \text{Re} t^2 > 0 \). The asymptotic form of (A.3) to the left of the point \( y_1 \) is obtained by the method of steepest descents. In this connection, the saddle points are determined from the equation \( \Delta y + (t^2 + k_1 t^4) = 0 \), from which it follows that
\[ t_{\text{Ai}, \text{Bi}} = \pm i(k_0 \pm \sqrt{-\Delta y})^k. \]

The choice of contours and the saddle points are shown in Fig. 9. After simple calculations we obtain asymptotic expressions for the two linearly independent solutions of Eq. (A.1) in the region \( y < y_1 \):
\[ v \approx C \left[ \exp \left\{ \frac{i \pi}{4} + \frac{i}{h} \int p \, dy' \right\} + \exp \left\{ \frac{i 3 \pi}{4} + \frac{i}{h} \int p \, dy' \right\} \right], \]
\[ v \approx C \left[ \exp \left\{ \frac{i 3 \pi}{4} - \frac{i}{h} \int p \, dy' \right\} + \exp \left\{ \frac{i \pi}{4} - \frac{i}{h} \int p \, dy' \right\} \right]. \]

By comparing formulas (A.4) with the quasiclassical solution (A.2) we obtain two relations for the coefficients \( A_1 \) and \( B_1 \).

\[ \text{Fig. 9} \]

Equations which are not sufficient for the determination of \( A_1 \) and \( B_1 \) follow from the boundary conditions on the surface of the metal. Having written the solution of Eq. (A.1) near \( y = y_0 \) in the form
\[ v(y) = A \exp \left\{ \frac{\pi}{4} + \int p \, dy' \right\} - \lambda \exp \left\{ \frac{3 \pi}{4} + \int p \, dy' \right\} + v_n, \]
where \( v_n \) satisfies the equation
\[ v_n'' + a(y - y_0) v_n = 0 \]  \hspace{1cm} (A.5)
and vanishes at the boundary of the sample, we also guarantee, with the necessary degree of accuracy, fulfillment of the boundary condition \( u(0) = 0 \), as is easy to verify. Here we have introduced the notation
\[ a = - \frac{2 m \sigma^2 (y_0)}{h^2} = \frac{\Delta^2}{\mu_0} > 0. \]

The solution of Eq. (A.5) of interest to us is obviously a linear combination of Airy functions. Matching the asymptotic form of the thus obtained exact solution with the wave function (A.2), we obtain two more relations between \( A_1 \) and \( B_1 \). The system of equations for the determination of \( A_1 \) and \( B_1 \) has a nontrivial solution provided

\[ S = 2 \pi A (n - 1/2 + \varphi / \pi), \]  \hspace{1cm} (A.6)

where
\[ S = 2 \int \int p \, dy' \int p \, dy, \quad t \varphi = \sin (\pi/12) - \lambda (y_0) \cos (\pi/12), \]
\[ \lambda (y_0) = - J_n (y_0) / J_{n+1} (y_0) \]

Let us consider a few special cases. Let \( y_0 \) be positive and large; then by using the asymptotic form of the Bessel functions one can easily obtain the result \( \varphi = \pi / 4 \), from which it follows that for trajectories of type \( b \) on Fig. 8 we have \( S = 2 \pi (n - 1) \) (compare with \( \varphi \)). Similarly for trajectories of type \( a \) shown in Fig. 8, that is, if \( y_0 < 0 \) and large in absolute value, we find that \( S = 2 \pi (n - 1/2) \). We note that for an “ordinary” particle in the case of two turning points on the classical phase trajectory \( \gamma = 1/2 \) (for a single turning point \( \gamma = 1/4 \)).

Thus we see that in the Bohr quantization rules the value of the phase is determined by the evenness (or oddness) of the number of turning points both for the excitations in a superconductor as well as for “ordinary” particles and quasiparticles. For an arbitrary number of turning points the same statement is easily proved by induction.

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9) An error was noted in [4], which was made in [7] in connection with writing down the general form of the transition matrix relating the coefficients in the wave function to the right and to the left of the turning points. This remark is based on a misunderstanding: the author of article [4] ignored the fact that the matrices in [2] and [4] differ only by an interchange of the rows, i.e., the components of the corresponding vector are relabeled.

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10. R. Courant and D. Hilbert, Methods of Mathematical Physics, John Wiley & Sons, Inc., 1953 (Vol. 1) and 1962 (Vol. 2).