Measurement Theory in Lax-Phillips Formalism

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Abstract. It is shown that the application of Lax-Phillips scattering theory to quantum mechanics provides a natural framework for the realization of the ideas of the Many-Hilbert-Space theory of Machida and Namiki to describe the development of decoherence in the process of measurement. We show that if the quantum mechanical evolution is pointwise in time, then decoherence occurs only if the Hamiltonian is time-dependent. If the evolution is not pointwise in time (as in Liouville space), then the decoherence may occur even for closed systems. These conclusions apply as well to the general problem of mixing of states.

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1. Introduction

Many attempts have been made for explaining quantum measurements [1]. Recently, Machida and Namiki have proposed a measurement theory called Many-Hilbert-Space (MHS) theory [2,3,4]. They observed that the explanation of quantum mechanical measurements only requires decoherence of the wave function, which can be formulated dynamically by extending the framework of quantum mechanics. In their theory, the key role is played by a direct integral space of continuously many Hilbert spaces and a continuous superselection rule. The direct integral space structure is assigned only to the measurement apparatus reflecting its macroscopic nature, while the observed system remains to be described by a single Hilbert space. Various situations such as double slit experiments and negative result experiments have been investigated in the framework of the MHS theory [3,4].

A direct integral space of continuously many Hilbert spaces appears naturally in the quantum version of the Lax-Phillips theory [5] introduced by Flesia, Piron and Horwitz [6,7]. In this approach, the direct integral space is introduced in order to allow the generator of motion to have a spectrum over the whole real axis, which is a necessary condition of the application of the Lax-Phillips theory. Contrary to the MHS theory, the direct integral space structure is assigned to every system irrespective to its size. Therefore, the Flesia-Piron-Horwitz approach may provide a measurement theory which inherits the desired features of the MHS theory and which does not need the clear-cut boundary between the system and the apparatus. In this paper, we show that, in fact, the quantum Lax-Phillips theory as constructed by Flesia and Piron admits the possibility of dynamical decoherence of states, and therefore may be used for the description of the measurement process. Futhermore, we show that using an extension of the Flesia-Piron form of the theory, which was shown [8] to be necessary for achieving the full structure of Lax-Phillips theory (i.e., non-trivial S-matrix), decoherence may occur even for *closed* (homogenous) systems.

We first introduce the Lax-Phillips description of quantum mechanics starting from the Schrödinger representations. Consider a quantum mechanical system with a Hilbert space $\mathcal{H}$. In the Lax-Phillips formalism introduced in refs. [6,7] the state space is described by a direct integral of (isomorphic) copies $\mathcal{H}_t$ of the Hilbert space $\mathcal{H}$, indexed by t

$$\overline{\mathcal{H}} \equiv \int_{\oplus} \mathcal{H}_t dt. \tag{1}$$

Scalar products in $\bar{\mathcal{H}}$ have the form

$$(f, g) = \int (f_t, g_t)_{\mathcal{H}} dt, \tag{2}$$

and the norm squared in

$$\| f \|_{\bar{\mathcal{H}}}^2 = \int \| f_t \|_{\bar{\mathcal{H}}}^2 dt. \tag{3}$$

We assume the existence of a unitary group of evolution operators parameterized by the laboratory time τ (which is not a dynamical variable of the system, but only a parameter) $U(\tau)$ in $\mathcal{H}$, with an infinitesimal generator K.

In the Flesia-Piron-Horwitz approach [6,7], the evolution $U(\tau)$ is expressed by an operator $W_t(\tau)$ local in the index t:

$$(U(\tau)\psi)_{t+\tau} \equiv \psi_{t+\tau}^{\tau} = W_t(\tau)\psi_t \tag{4}$$

with the generator

$$(K\psi)_t = -i\partial_t \psi_t + H(t)\psi_t, \tag{5}$$

where $H(t)$ corresponds to the Hamiltonian in the usual Hilbert space $\mathcal{H}$.

It was recently shown [8] that in order to achieve the full structure of the Lax-Phillips theory (i.e., a non-trivial S-matrix which can be related to the Lax-Phillips semigroup), one must consider a more general evolution law of the form

$$(U(\tau)\psi)_{t+\tau} = \int_{-\infty}^{+\infty} W_{t,t'}(\tau)\psi_{t'}dt'; \tag{6}$$

the generator of the evolution in this case takes the form

$$(K\psi)_t = -i\partial_t\psi_t + \int_{-\infty}^{+\infty} \kappa_{t,t'}\psi_{t'}dt', \tag{7}$$

We remark that the structure of the evolution in the Liouville space [10] has precisely this form [8], if we take the t-representation to correspond to an operator conjugate to the unperturbed Liouvillian. We shall carry out our study of decoherence here in the framework of the ordinary (direct integral) Hilbert space; the application of these ideas to the Liouville space will be considered elsewhere.

Any time-dependent observable $A(t)$ defined in the usual quantum Hilber space can be naturally lifted to the direct integral space $\bar{\mathcal{H}}$ as follows

$$(\hat{A}\psi)_t = A(t)\psi_t \tag{8}$$

As a natural generalization, the expectation value of any observable $\hat{A}$ in the direct integral space is defined by

$$\langle\hat{A}\rangle_\psi = \frac{(\psi, \hat{A}\psi)_{\bar{\mathcal{H}}}}{(\psi, \psi)_{\bar{\mathcal{H}}}} = \frac{\int dt(\psi_t, A(t)\psi_t)_{\bar{\mathcal{H}}}}{\int dt(\psi_t, \psi_t)_{\bar{\mathcal{H}}}}. \tag{9}$$

In what follows, we will formulate and study decoherence in this framework.

2. Pure and Mixed States

We wish to show now that a pure state in the larger Hilbert space (which we will refer to as a pure Lax-Phillips state), can represent both pure and mixed states in the usual sense.

Practically, almost all measurement processes correspond to measurements of observables which are time-dependent in the Schrödinger picture, such as a projection operator to a given subspace or the asymptotic Heisenberg variables of a scattering process. The lift of such operators to $\bar{\mathcal{H}}$ is t-independent also. Therefore, if two different states give the same expectation value for a complete set of time independent observables, the two states are indistinguishable. In this sense, we define the following:

1. A Lax-Phillips state ψ is called "pure-like" if there exists a pure state ϕ_0 in the original Hilbert space $\mathcal{H}$ such that

$$\langle\hat{A}\rangle_\psi = \frac{(\phi_0, A\phi_0)}{(\phi_0, \phi_0),} \tag{10}$$

holds for any time-independent observable A on the original Hilbert space $\mathcal{H}$.

2. A Lax-Phillips state ψ is called "mixed-like" if there exists a density matrix ρ_0 in the original Hilbert space $\mathcal{H}$ for which $Tr\rho_0^2 < 1$, such that

$$\langle\hat{A}\rangle_\psi = Tr(A\rho_0), \tag{11}$$

holds for any time-independent observable A on the original Hilbert space $\mathcal{H}$.

For a time-independent obervable A on $\mathcal{H}$, Eq. (9) gives

$$\langle\hat{A}\rangle_\psi = \frac{\int dt(\psi_t, A\psi_t)_{\mathcal{H}}}{\int dt(\psi_t, \psi_t)_{\mathcal{H}}} = Tr(\rho_\psi A), \tag{12}$$

where ρ_ψ is the density matrix on the original Hilbert space $\mathcal{H}$ associated with the Lax-Phillips state ψ, defined as

$$\rho_\psi = \frac{1}{\mathcal{N}} \int dt|\psi_t\rangle\langle\psi_t|; \qquad \mathcal{N} = \int dt(\psi_t, \psi_t)_{\mathcal{H}}. \tag{13}$$

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Therefore, if ρ_ψ is pure, the Lax-Phillips state ψ is pure-like, and if ρ_ψ is mixed the Lax-Phillips state ψ is mixed-like.

The state $\psi = \{\psi_t\}$ is pure-like if and only if there is a normalized state $\psi_0 \in \mathcal{H}$ and a scalar function $f(t)$ satisfying

$$\int dt |f(t)|^2 = 1,$$

such that

$$\psi_t = \psi_0 f(t). \tag{14}$$

Indeed (14) gives $\mathcal{N} = 1$ and

$$\rho_\psi = \int dt |\psi_t\rangle\langle\psi_t| = \int dt |f(t)|^2 |\psi_0\rangle\langle\psi_0| = |\psi_0\rangle\langle\psi_0|. \tag{15}$$

In general

$$Tr\rho_\psi^2 = \Sigma^i \int\int dt dt' \langle\psi_i|\psi_t\rangle\langle\psi_t|\psi_{t'}\rangle\langle\psi_{t'}|\psi_i\rangle$$
$$= \int\int dt dt' |\langle\psi_t|\psi_{t'}\rangle|^2, \tag{16}$$

where $\{\psi_i\}$ is a complete orthonormal set in $\mathcal{H}$. By the Schwartz inequality, unless ψ_t is proportional to $\psi_{t'}$, i.e., of the form (14),

$$|\langle\psi_t|\psi_{t'}\rangle|^2 < \|\psi_t\|_\mathcal{H}^2 \|\psi_{t'}\|_\mathcal{H}^2, \tag{17}$$

and hence $Tr\rho_\psi^2 < 1$.

3. Decoherence in the Flesia-Piron Approach

In this and the next sections, we discuss the possibility of decoherence, or the evolution from pure-like to mixed-like states. Here we treat the problem in the framework of the Flesia-Piron approach.

First we consider the Schrödinger evolution for a time-dependent Hamiltonian. The solution of the time-dependent Schrödinger equation can always be written formally as $\psi_t = U(t, t')\psi_{t'}$, where $U(t, t')$ satisfies the chain property $U(t, t')U(t', t'') = U(t, t'')$, and can be expressed in terms of the integral of a time-ordered product. We define $W_t(\tau) = U(t + \tau, t)$, and lift the evolution to $\bar{\mathcal{H}}$ as follows [9]

$$\psi_{t+\tau}^\tau = W_t(\tau)\psi_t, \tag{18}$$

where $W_t(\tau)$ is given by (T implies the time-ordered product)

$$W_t(\tau) = T(e^{-i\int_t^{t+\tau} H(t')dt'}). \tag{19}$$

For this kind of time-evolution we obtain

$$\rho_\psi(\tau) = \frac{1}{\mathcal{N}(\tau)} \int dt |(U(\tau)\psi)_t\rangle\langle(U(\tau)\psi)_t|$$
$$= \frac{1}{\mathcal{N}} \int dt |(U(\tau)\psi)_{t+\tau}\rangle\langle(U(\tau)\psi)_{t+\tau}|$$
$$= \frac{1}{\mathcal{N}} \int dt |w_t(\tau)\psi_t\rangle\langle W_t(\tau)\psi_t|, \tag{20}$$

where we have used the fact that the normalization constant is time-independent (which follows from the unitarity of $U(\tau)$). For the pure-like state introduced in (14), we then have

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$$\rho_{\psi_P}(\tau) = \int dt |f(t)|^2 W_t(\tau) |\psi_0\rangle \langle \psi_0| W_t^\dagger(\tau). \tag{21}$$

It follows from (16) that this state is "mixed-like" if $W_t(\tau)|\psi_0\rangle$ is not proportional to $W_{t'}(\tau)|\psi_0\rangle$ for $t \neq t'$ (the set for which $t = t'$ is of zero measure).

The evolution operator $W_t(\tau)$ does not depend on t if and only if the system is invariant to translations along the t-axis, i.e., the Hamiltonian $H(t)$ is time-independent. In this case $W_t(\tau) = W(\tau) = e^{-iH\tau}$ and

$$\rho_{\psi P}(\tau) = \int dt |f(t)|^2 W(\tau)|\psi_0\rangle \langle \psi_0| W^\dagger(\tau) = W(\tau)|\psi_0\rangle \langle \psi_0| W^\dagger(\tau) \tag{22}$$

is again a pure-state. In other words, in the Flesia-Piron approach if the Hamiltonian does not depend on time explicitly, a pure-like state remains pure-like, and there arises no decoherence. On the other hand, if the Hamilitionian depends on time explicitly, the states, in general, cannot maintain their purity and decoherence takes place. As we shall see in a concrete example, the degree of decoherence depends not only on the time-dependence of the Hamiltonian, but also on the initial states.

4. Decoherence in Closed System

As shown in the previous section, the original Flesia-Piron approach may allow decoherence only for systems with explicit time dependence, i.e., for open systems. This is not so satisfactory as the system plus apparatus can always be seen as a closed system, where decoherence takes place without external disturbances. As briefly explained in the Introduction, the Flesia-Piron approach has been recently generalized by introducing an interaction which is non-local on the time axis (cf. eqs. (6) and (7)) [8]. As we shall see now, the generalization provides a possibility of decoherence even for closed systems.

Choosing, in particular, a kernel κ of the form (cf. eq. (7))

$$\kappa_{t,t'} = \kappa_{t-t'}, \tag{23}$$

it follows that (here, $-i\partial_t$ stands for the operator on $\bar{\mathcal{H}}$ which is represented as a derivative in the t-representation)

$$[\kappa, -i\partial_t] = 0. \tag{24}$$

Therefore, the system described by generator of this form is closed in the sense that it is invariant to translations on the time-axis, i.e.,

$$[K, -i\partial_t] = 0. \tag{25}$$

It is shown in the Appendix that this kind of interaction leads to an evolution operator of the form

$$W_{t,t'}(\tau) = \frac{1}{2\pi} \int e^{i(t-t')\sigma} e^{-i\kappa(\sigma)\tau} d\sigma = W_{t-t'}(\tau) \tag{26}$$

where $\kappa(\sigma)$ is the Fourier transform of $\kappa_{t-t'}$ with respect to $t - t'$ and is Hermitian.

Now, consider the most general form of pure state, $\psi_t = f(t)\psi_0$. If follows from (6) that the time evolution of such a state is

$$(\psi^\tau)_{t+\tau} = \int W_{t,t'}(\tau)\psi_0 f(t')dt'. \tag{27}$$

Since the evolution operators satisfy (from (26)) $W_{t,t'}(\tau) = W_{t-t'}(\tau)$, it follows that

$$(\psi^\tau)_{t+\tau} = \int W_{t-t'}(\tau)\psi_0 f(t')dt' =$$
$$= \int W_{t'}(\tau)\psi_0 f(t-t')dt'. \tag{28}$$

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This corresponds, for every t, to a superposition of the states $W_{t'}(\tau)\psi_0$, but, in general, for each t, the weights are different, and we conclude that the state may be mixed by the evolution (cf. the arguments below (16)). The purity of the state will be conserved if and only if all the states $W_{t'}(\tau)\psi_0$ are the same up to a factor which is a function of t' (and τ; the discussion which follows is, however, for each τ). We shall now prove that this occurs for any ψ_0 if and only if $\kappa_{t-t'} = \kappa\delta(t-t')$, where κ is some constant operator.

Let us assume that

$$W_t(\tau)\psi_0 = \alpha_t\psi_1 \tag{29}$$

for any arbitrary ψ_0 and corresponding ψ_1. Let $\{\phi_n\}$ be a complete orthonormal set in $\bar{\mathcal{H}}$; then for each τ,

$$W_t(\tau)\phi_n = \alpha_t\psi_n = \alpha_t\sum_n \beta_{mn}\phi_m, \tag{30}$$

and therefore

$$(\phi_m, W_t(\tau)\phi_n) = \beta_{mn}\alpha_t. \tag{31}$$

Hence,

$$W_t(\tau) = \alpha_t W(\tau), \tag{32}$$

where

$$(\phi_m, W(\tau)\phi_n) = \beta_{mn}. \tag{33}$$

Taking the Fourier transform of (32) one obtains

$$\tilde{W}_\sigma(\tau) = \tilde{\alpha}(\sigma)W(\tau). \tag{34}$$

On the other hand, from (26) it follows that

$$\tilde{W}_\sigma(\tau) = e^{-i\kappa(\sigma)\tau}. \tag{35}$$

We show first that $W(\tau)$ has an inverse. It is shown elsewhere [8] that the evolution operators satisfy the relation

$$\int W_{t,t''}(\tau)W_{t',t''}(\tau)^\dagger dt'' = \delta(t-t'). \tag{36}$$

Using (32) if follows from (36) that

$$\int \alpha_{t-t''}\alpha^*_{t'-t''}dt''W(\tau)W(\tau)^\dagger = \delta(t-t'), \tag{37}$$

and, by integrating it with respect to t,

$$\lambda W(\tau)W(\tau)^\dagger = 1 \tag{38}$$

with

$$\lambda = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{\infty} \alpha_{t-t''}\alpha^*_{t''-t'}dt'',$$

i.e.,

$$W^{-1}(\tau) = \lambda W(\tau)^\dagger. \tag{39}$$

Hence, W^{-1} exists. Then, from (34) and (35) it follows that

$$\tilde{W}_\sigma(\tau_1)\tilde{W}_\sigma(\tau_2)^{-1} = e^{-i\kappa(\sigma)(\tau_1-\tau_2)} = W(\tau_1)W(\tau_2)^{-1}, \tag{40}$$

independently of σ; hence

$$\kappa(\sigma) = \text{const.} \Rightarrow \kappa_{t-t'} = \kappa\delta(t-t'). \tag{41}$$

Thus, we realize that pure states remain pure if and only if condition (41) is satisfied, which is exactly the case of a time-independent, pointwise Hamiltonian.

The density matrix corresponding to the Lax-Phillips state (28) is given by

$$\rho_\psi = \int dt |\psi^\tau_{t+\tau}\rangle\langle\psi^\tau_{t+\tau}| = \int d\sigma e^{i\kappa(\sigma)\tau} |\psi_0\rangle\langle\psi_0| e^{-i\kappa(\sigma)\tau}, \tag{42}$$

with

$$P(\sigma) = \frac{1}{2\pi}\left|\int dt e^{-it\sigma} f(t)\right|^2. \tag{43}$$

Since $P(\sigma) \geq 0$ and $\int d\sigma P(\sigma) = 1$, the expression (42) implies that the density matrix ρ_ψ is a convex combination of pure state evolutions by a "Hamiltonian $\kappa(\sigma)$" weighted by a "probability $P(\sigma)$". Therefore, it is, in general, a mixed state. Moreover, the evolution (42) is formally the same as that appears in the MHS theory (cf. eq. (5.17) of [4]). We therefore see that a generalized evolution of the form (6) may lead to mixing of pure states for closed systems and that the generalized formulation may possess all the aspects of the MHS theory.

5. Examples

In order to see the details of the decoherence processes, we study two simple examples corresponding to the time-local evolution (4) and the time-nonlocal evolution (6) respectively.

First, let us consider a system described by the following Hamiltonian

$$H(t) = -\frac{\Omega_0}{2}\Sigma_z + \frac{\Omega}{2}[\Sigma_+ e^{i\Omega_0 t} + \Sigma_- e^{-i\Omega_0 t}], \tag{44}$$

where Σ_i are the Pauli matrices:

$$\Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \Sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The Hilbert space for this model is the two dimensional complex space $\mathcal{H} = \mathbf{C}^2$. It is easy to derive the evolution operator $W_t(\tau)$ corresponding to the Hamiltonian $H(t)$. One obtains

$$W_t(\tau) = u(\tau)\{\cos\frac{\Omega}{2}\tau - i\sin\frac{\Omega}{2}\tau(\Sigma_+ e^{i\Omega_0 t} + \Sigma_- e^{-i\Omega_0 t})\}, \tag{45}$$

where the operator $u(\tau)$ is given by

$$u(\tau) = \begin{pmatrix} \exp(i\frac{\Omega_0}{2}\tau) & 0 \\ 0 & \exp(-i\frac{\Omega_0}{2}\tau). \end{pmatrix} \tag{46}$$

The direct integral space for this model is given by $L^2(-\infty, \infty; \mathbf{C}^2)$. We wish to study now the time evolution of pure-like state given by Eq. (14). From Eqs. (18) and (45) we have

$$\rho_{\psi_P} = u(\tau) \int dt |f(t)|^2 \{\cos(\frac{\Omega}{2}\tau) - i\sin(\frac{\Omega}{2}\tau)(\Sigma_+ e^{i\Omega_0 t} + \Sigma_- e^{-i\Omega_0 t})\}|\psi_0\rangle$$

$$\times \langle\psi_0|\{\cos(\frac{\Omega}{2}\tau) + i\sin(\frac{\Omega}{2}\tau)(\Sigma_+ e^{i\Omega_0 t} + \Sigma_- e^{i-\Omega_0 t})\}u^\dagger(\tau)$$

$$= u(\tau)\Big[\cos^2(\frac{\Omega}{2}\tau)|\psi_0\rangle\langle\psi_0| + \sin^2(\frac{\Omega}{2}\tau)\Sigma_+|\psi_0\rangle\langle\psi_0|\Sigma_-$$

$$+ \sin^2(\frac{\Omega}{2}\tau)\Sigma_-|\psi_0\rangle\langle\psi_0|\Sigma_+ \tag{47}$$

$$+ \{iF(\Omega_0)\cos(\frac{\Omega}{2}\tau)\sin(\frac{\Omega}{2}\tau)(|\psi_0\rangle\langle\psi_0|\Sigma_+ - \Sigma_-|\psi_0\rangle\langle\psi_0|)$$

$$+ \sin^2(\frac{\Omega}{2}\tau)F(2\Omega_0)\Sigma_+|\psi_0\rangle\langle\psi_0|\Sigma_+ + h.c.\}\Big]u^\dagger(\tau),$$

where $F(\omega)$ is the Fourier transform of $|f(t)|^2$

$$F(\omega) \equiv \int dt |f(t)|^2 e^{i\omega t}. \tag{48}$$

As an example, suppose we take $|f(t)|^2$ to be the Gaussian form

$$|f(t)|^2 = \frac{1}{2\sqrt{\pi}\triangle} \exp\left[-\frac{(t-t_0)^2}{4\triangle^2}\right], \tag{49}$$

then we obtain

$$F(\omega) = e^{i\omega t_0} e^{-(\omega\triangle)^2}. \tag{50}$$

In order to study the decoherence quantitatively, we introduce a degree of decoherence ε (which is different from the one introduced in [3,4])

$$\varepsilon = Tr\rho_{\psi_P}^2 - 1. \tag{51}$$

Obviously, ε represents a "distance" from pure states (a state with $\varepsilon \neq 0$ is mixed and a state with $\varepsilon = 0$ is pure). For the state (47) with the Gaussian form of $f(t)$ given by (49), a tedious but straightforward calculation gives

$$\varepsilon = 2g(\Omega_0)\sin^2\frac{\Omega\tau}{2}\{\sin^2\frac{\Omega\tau}{2}|\langle\Sigma_+\Sigma_-\rangle|^2(|F(\Omega_0| + 1)(|F(\Omega_0)|^2 + 1)(|F(2\Omega_0)| + 1)$$

$$+ \cos^2\frac{\Omega\tau}{2}(1 - 2\langle\Sigma_+\rangle\langle\Sigma_-\rangle)(|F(\Omega_0)| + 1) + \Big[\cos^2\frac{\Omega\tau}{2}\langle\Sigma_+\rangle^2 F(\Omega_0)^2(|F(\Omega_0)| + 1) \tag{52}$$

$$+ i\sin\frac{\Omega\tau}{2}\cos\frac{\Omega\tau}{2}\langle\Sigma_-\rangle\langle\Sigma_z\rangle F(\Omega_0)^*(|F(\Omega_0)| + 1)(|F(\Omega_0)|^2 + 1) + c.c.\Big]\},$$

where $\langle\ldots\rangle$ stands for the average with respect to the state ψ_0 and the function $g(\Omega_0) \equiv |F(\Omega_0)| - 1$ describes the initial state dependence of the degree of decoherence. The deviation of the state ρ_{ψ_P} from pure states can also be seen directly on the operator level. Indeed, it can be rewritten as

$$\rho_{\psi_P} = u(\tau)\tilde{W}(\tau)|\psi_0\rangle\langle\psi_0|\tilde{W}^\dagger(\tau)u^\dagger(\tau)$$

$$+ g(\Omega_0)\sin\frac{\Omega(\tau)}{2}u(\tau)\{i\ e^{i\Omega_0 t_o}\cos\frac{\Omega(\tau)}{2}(|\psi_0\rangle\langle\psi_0|\Sigma_+ - \Sigma_+|\psi_0\rangle\langle\psi_0|) \tag{53}$$

$$+ \sin\frac{\Omega(\tau)}{2}(|F(\Omega_0)| + 1)(|F(\Omega_0)|^2 + 1)e^{2i\Omega_0 t_0}\Sigma_+|\psi_0\rangle\langle\psi_0|\Sigma_+ + h.c.\}u^{\dagger(\tau)},$$

where $\tilde{W}(\tau)$ is a unitary operator given by

$$\tilde{W}(\tau) = \cos\frac{\Omega(\tau)}{2} - i\sin\frac{\Omega(\tau)}{2}(e^{i\Omega_0 t_0}\Sigma_+ + e^{-i\omega_0 t_0}\Sigma_-). \tag{54}$$

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Strictly speaking, as the degree of coherence is different from zero ($\varepsilon \neq 0$), decoherence takes place irrespective of the value of $\triangle(\neq 0)$. However, if g and thus the degree of decoherence ε are very small, the state ρ_{ψ_P} corresponds to an almost pure state. In short, we find for the Gaussian example in the Flesia-Piron approach, that when the initial state is well localized on the t-axis compared with the time scale of the change of the Hamiltonian, i.e., $\Omega_0 \triangle << 1$, the state ρ_{ψ_P} remains practically pure. Otherwise, decoherence takes place.

Next we consider a system with a time-nonlocal interaction:

$$\kappa_{t,t'} = \kappa_{t-t'} = \frac{\Omega}{2}(\Sigma_+ \delta(t - t' + t_d) + \Sigma_- \delta(t - t' - t_d), \tag{55}$$

which leads to

$$\kappa(\sigma) = \int d(t - t') e^{-i(t-t')\sigma} \kappa_{t-t'} = \frac{\Omega}{2}(\Sigma_+ e^{it_d \sigma} + \Sigma_- e^{-it_d \sigma}), \tag{56}$$

and, thus,

$$e^{-i\kappa(\sigma)\tau} = \cos\frac{\Omega\tau}{2} - i\sin\frac{\Omega\tau}{2}(e^{i\sigma t_d}\Sigma_+ + e^{-i\sigma t_d}\Sigma_-). \tag{57}$$

Because of (42), we then have

$$
\begin{aligned}
\bar{\rho}_{\psi_P} = {} & \int d\sigma P(\sigma)\{\cos\frac{\Omega\tau}{2} - i\sin\frac{\Omega\tau}{2}(e^{i\sigma t_d}\Sigma_+ + e^{-i\sigma t_d}\Sigma_-)\}|\psi_0\rangle \\
& \times \langle\psi_0|\{\cos\frac{\Omega\tau}{2} + i\sin\frac{\Omega\tau}{2}(e^{i\sigma t_d}\Sigma_+ + e^{-i\sigma t_d}\Sigma_-)\} \\
= {} & \cos^2\frac{\Omega\tau}{2}|\psi_0\rangle\langle\psi_0| + \sin^2\frac{\Omega\tau}{2}\Sigma_+|\psi_0\rangle\langle\psi_0|\Sigma_- \\
& + \sin^2\frac{\Omega\tau}{2}\Sigma_-|\psi_0\rangle\langle\psi_0|\Sigma_+ \\
& + \{i\bar{F}(t_d)\sin\frac{\Omega\tau}{2}\cos\frac{\Omega\tau}{2}(|\psi_0\rangle\langle\psi_0|\Sigma_+ - \Sigma_+|\psi_0\rangle\langle\psi_0|) \\
& + \sin^2\frac{\Omega\tau}{2}\bar{F}(2t_d)\Sigma_+|\psi_0\rangle\langle\psi_0|\Sigma_+ + \text{h.c.}\},
\end{aligned} \tag{58}
$$

where $\bar{F}(t)$ is the Fourier transform of the "probability density" $P(\sigma)$,

$$\bar{F}(t) \equiv \int d\sigma P(\sigma) e^{i\sigma t}. \tag{59}$$

From eqs. (47) and (58), we find that the density operators ρ_{ψ_P} and $\bar{\rho}_{\psi_P}$ have an identical form except for the unitary operator $u(\tau)$ and the fact that F in (47) is replaced by $\bar{F}$ in (58). Thus, their decoherence properties are the same. For the Gaussian form (49) of $F(t)$, we have

$$\bar{F}(t) \ \exp(-\frac{t^2}{16\triangle^2}) = |F(t/(4\triangle^2))|. \tag{60}$$

Therefore, in this case, the degree of decoherence ε and the decomposition into purity preserving and purity nonpreserving terms of the density operator $\bar{\rho}_{\psi_P}$ can be obtained from the corresponding expressions (52) and (53) for ρ_{ψ_P} by replacing Ω_0 with $t_d/(4\triangle^2)$ and dropping $e^{i\Omega_0 t_0}, e^{2i\Omega_0 t_0}$ and $u(\tau)$. We then find that when the initial state is well delocalized on the t-axis compared with the non-locality of the interaction, i.e., $t_d/\triangle << 1$, the state ρ_{ψ_P} remains practically pure.

Interestingly enough, the dependence of the degree of decoherence ε upon the initial spread $\triangle$ on the t-axis for the first example is opposite to that for the second one. The difference can be understood as follows: In the first example, as the Hamiltonian changes in time, the state with larger spread cannot follow the change of the Hamiltonian and loses its coherence. Contrarily, in the second example, the state can preserve its purity only when it has enough spread not to feel the non-locality of the interaction.

In short, we have shown that the details of the decoherence processes depend not only on the dynamics of the systems but also on the initial conditions.

6. Conclusion

We have that the Lax-Phillips theory provides a description of the quantum states which admits the possibility of decoherence for time-dependent Hamiltonian systems, and even for systems which are closed (but not of Hamiltonian form in the original Hilbert space). As we have seen in the concrete examples, the degrees of decoherence depends not only on the structure of the system (i.e., the generator of motion), but also on the initial conditions. In other words, a given system may behave as a (almost) pure quantum system (coherent time evolution) or as a system plus an apparatus (incoherent time evolution) depending on the initial conditions. Intermediate situations are also possible. Therefore, the Many-Hilbert-Space theory can be formulated naturally in this framework, and it is not necessary to specify the limit between the system and the measuring apparatus. As we have remarked, the relation between the singularities of the S-matrix and the spectrum of the generator of the semigroup can be obtained only from such a general evolution. We therefore see that the origin of irreversibility may be found in such structures.

Appendix

The basic equation which relates the generator κ to the evolution operators is as follows:

$$i\partial_\tau W_{t,t'}(\tau) = \int \kappa_{t+\tau,t''} W_{t'',t'}(\tau) dt''. \tag{A1}$$

Let us take the Fourier transform of (A1) with respect to t and t'.

$$i\partial_\tau W_{\sigma,\sigma'}(\tau) = \int e^{i\sigma\tau} \kappa_{\sigma,\sigma''} e^{-i\sigma''\tau} W_{\sigma'',\sigma'}(\tau) d\sigma''. \tag{A2}$$

For closed systems, in which $\kappa_{t,t'} = \kappa_{t-t'}$, one obtains

$$\kappa_{\sigma,\sigma'} = \tilde{\kappa}(\sigma)\delta(\sigma - \sigma'). \tag{A3}$$

Using (A3) in (A2), one obtains

$$i\partial_\tau W_{\sigma,\sigma'}(\tau) = \tilde{\kappa}(\sigma) W_{\sigma,\sigma'}(\tau)$$

from which follows

$$W_{\sigma,\sigma'}(\tau) = e^{-i\tilde{\kappa}(\sigma)\tau}\delta(\sigma - \sigma'). \tag{A4}$$

Taking the inverse transform of (A4) we get

$$\begin{aligned} W_{t,t'}(\tau) &= \frac{1}{2\pi} \int d\sigma d\sigma' e^{i\sigma t} e^{-i\tilde{\kappa}(\sigma)\tau} \delta(\sigma - \sigma') e^{-i\sigma' t'} = \\ &= \frac{1}{2\pi} \int d\sigma e^{i\sigma(t-t')} e^{-i\tilde{\kappa}(\sigma)\tau} \end{aligned} \tag{A5}$$

which depends only in $t - t'$.

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