RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 49, NUMBER 2 FEBRUARY 1994

Suppression of chaos, quantum resonance, and statistics of a nonintegrable system

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(Received 23 August 1993)

The quantum motion of a periodically two-sided kicked rotator is studied. It is shown that while this system is classically chaotic, its quantized version is exactly periodic for a suitable choice of the relevant parameters, and thus not only the chaotic diffusion behavior is suppressed in the course of quantization, but also the sensitivity to initial conditions. The statistical properties of an ensemble of such systems are also investigated; the distribution of the probability density is shown to obey the Rayleigh statistics.

PACS number(s): 05.45.+b

Much attention has been paid in recent years to the study of the effects of chaos in classical Hamiltonian systems on their quantum counterparts [1,2]. The signs of the chaotic behavior in these quantum systems may be used to define the notion of quantum chaos, which is still unclear. The quantum version of the kicked rotator, traditionally one of the favorite model systems in the theory of classical chaos [1], has been extensively studied in this connection [2,3]. The classical kicked rotator depends on a single parameter K, the dimensionless strength of the kick. For each value of K the motion is chaotic or regular depending on the initial conditions. For small K the chaotic regions are isolated and are separated by Kolmogorov-Arnold-Moser (KAM) trajectories, and consequently the motion is bounded. For $K=K_c=0.971\,64\ldots$ the last of these trajectories disappear, and for $K>K_c$ chaotic diffusion in angular momentum L takes place, i.e., $L^2\sim t$ for large t.

It has been found [3] that the quantum-kicked-rotator model can be mapped into Anderson's problem of motion of a quantum particle in a one-dimensional lattice in the presence of a static diagonal disorder. Since all the eigenfunctions of a one-electron random Hamiltonian are exponentially localized in space [4] and therefore the electronic diffusion coefficient and the electron mobility vanish at zero temperature, it follows from this mapping that the quantum dynamical system is localized in angular-momentum space and hence can reach only a limited number of angular-momentum states in the course of its time evolution. This in turn implies quasiperiodicity and thus boundedness and recurrence of the energy in time [2], and is in contrast to the chaotic diffusion behavior obtained for the corresponding classical system for which the motion is an unrestricted random walk in angular-momentum space [5].

In this Rapid Communication we consider a slightly different model, for which the classical behavior is chaotic in an appropriate region of phase space, and show that quantization of this system leads to an exact periodicity of the wave function in time for suitable values of the relevant parameters. This behavior implies the complete suppression of the chaotic features, where not only the chaotic diffusion disappears but also the strong (exponential) dependence on initial conditions, and the time evolution of the expectation values is just the same as for integrable systems [9]. For the other region in parameter space, we use the ladder (diffusion) approximation introduced in the weak localization theory [7], to predict the statistics of an ensemble of such systems.

The model we discuss is a variation of the traditional-kicked-rotator model in which the driving term is given by a sequence of two-sided δ impulses, instead of one-sided kicks, i.e., the interaction term is given by

$$H_{\rm int}=\hat{k}V(\theta)\sum_{n=-\infty}^{\infty}\left[\delta\left[\frac{t}{T}-n\right]-\delta\left[\frac{t}{T}-\left[n+\frac{1}{2}\right]\right]\right]. \tag{1}$$

Perturbations of this kind were discussed in the context of the dissociation of molecules [6] and it has been shown that they may be used as an approximation to a sinusoidal driving term corresponding to an ac electromagnetic field. In fact, one has

$$\sum_{n=-\infty}^{\infty}\left[\delta\left[\frac{t}{T}-n\right]-\delta\left[\frac{t}{T}-\left[n+\frac{1}{2}\right]\right]\right]=4\sum_{n=1}^{\infty}\cos[(2n-1)\Omega t]\ , \tag{2}$$

where $\Omega=2\pi/T$. Thus, where one can neglect the effect of the larger frequencies, this interaction term is a good approximation for the ac field.

The Hamiltonian of the system is given by

$$H+H_0+H_{\rm int}\equiv\frac{L^2}{2I}+H_{\rm int}\ , \tag{3}$$

where $H_{\rm int}$ is given by Eq. (1). Hamilton's equations associated with this Hamiltonian are given by

$$d\theta/dt=L/I\ ,$$
$$dL/dt=-\hat{k}V'(\theta)\sum_{n=-\infty}^{\infty}\left[\delta\left[\frac{t}{T}-n\right]-\delta\left[\frac{t}{T}-\left[n+\frac{1}{2}\right]\right]\right], \tag{4}$$

where $V'(\theta)=\partial V(\theta)/\partial\theta$. Introducing dimensionless variables defined as

$$l=\frac{LT}{2I},\quad \tau=t/T,\quad k=\frac{\hat{k}T^2}{2I}\ , \tag{5}$$

we can write Eqs. (4) in the following form:

$$d\theta/d\tau=2l,$$
$$dl/d\tau=-kV'(\theta)\sum_{n=-\infty}^{\infty}\left[\delta(\tau-n)-\delta\left[\tau-\left[n+\frac{1}{2}\right]\right]\right]. \tag{6}$$

Integrating (6) over one period, one obtains the map

$$\theta_{\tau+1}=\theta_\tau+l_\tau-kV'(\theta_\tau+l_\tau)\ ,$$
$$l_{\tau+1}=l_\tau-kV'(\theta_\tau+l_\tau)+kV'(\theta_{\tau+1})\ . \tag{7}$$

This map, as well as the map obtained for the one-sided kicked rotator, exhibits chaotic features as one can clearly see from Figs. 1 and 2.

We now consider the quantized case, for which the system evolves in time as $\psi(t+T)=e^{-iHT/\hbar}\psi(t)$ where H is the Hamiltonian given by Eq. (3). Since the Hamiltonian is periodic in time, the evolution is determined by the Floquet operator, corresponding to the evolution of the system in one period. This Floquet operator e^{-iF} takes, in this case, the form

$$e^{-iF}=e^{-i\tilde{k}V(\theta)}e^{-iH_0T/2\hbar}e^{i\tilde{k}V(\theta)}e^{-iH_0T/2\hbar}\ , \tag{8}$$

where $\tilde{k}=\hat{k}T/\hbar=2Ik/\hbar T$. We then use the identity

$$e^{-iV}e^{W}e^{iV}=\exp(e^{-iV}We^{iV}) \tag{9}$$

in order to find that

$$e^{-iF}=e^{-i\tilde{H}_0T/2\hbar}e^{-iH_0T/2\hbar}\ , \tag{10}$$

where $\tilde{H}_0\equiv e^{-i\tilde{k}V(\theta)}H_0e^{i\tilde{k}V(\theta)}$. One sees that the operators H_0 and $\tilde{H}_0$ are related to each other by a unitary transformation and therefore they have the *same spectrum*. Let us denote the complete set of eigenfunctions of the operator H_0 as $\phi_n(\theta)$ and for $\tilde{H}_0$ as $\psi_n(\theta)$, where

$$\phi_n=\sum_m\alpha_{mn}\psi_m\ . \tag{11}$$

For an arbitrary initial condition $\psi(\theta,t)=\sum_n\beta_n\phi_n$, the wave function after one period is given by

$$\psi(t+T)=e^{-iF}\psi(t)$$
$$=\sum_{m,n}\beta_n\alpha_{mn}e^{-i(E_n+E_m)T/2\hbar}\psi_m\ . \tag{12}$$

The unperturbed Hamiltonian H_0, and its unitarily equivalent $\tilde{H}_0$, both have the property that

$$E_n=f(n)E_1\ , \tag{13}$$

where $E_1=\hbar^2/2I$ is the energy of the first excited state of H_0, and $f(n)=n^2$ takes only integer values. It is therefore clear that when

$$T=4\pi p\frac{\hbar}{E_1} \tag{14}$$

(where p is an integer) the wave function turns back to its initial value *for any step,* i.e., there is an *exact* periodicity of the wave function. Since in this case the Floquet operator e^{-iF} acts trivially, and $\psi(t)$ is arbitrary, it follows that the Floquet operator is equal to the identity operator, and therefore the only eigenvalue of this operator is 1, and the spectrum is discrete. This result implies that, if we take two different initial states ψ_1 and ψ_2, the (Hilbert space) distance between these states $\|\psi_1(t)-\psi_2(t)\|$ is continuous and periodic as a function of t (with the same frequency as the perturbation), and therefore, it is *bounded* by its maximum in the segment $[0,T]$. This, in turn, implies that for any observable of the system, differences in the initial states lead to bounded differences in the expectation values of this observable during the evolution of the system, in sharp contrast with the classical chaotic case (exponential sensitivity to initial conditions).

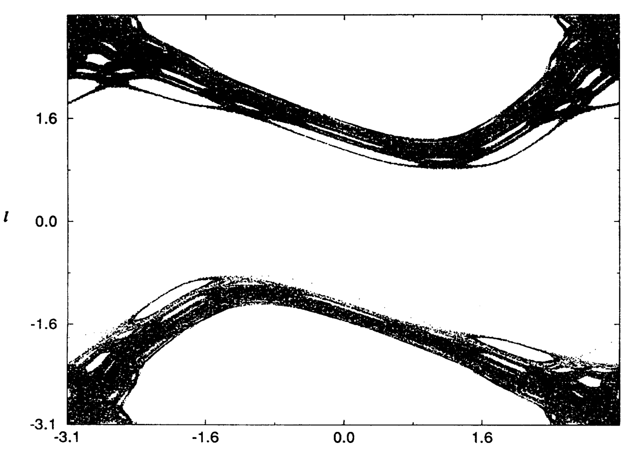

FIG. 1. Chaotic orbit for the classical map (7), with $V(\theta)=\cos(\theta)$ and $k=0.9$, corresponding to local chaos regime.

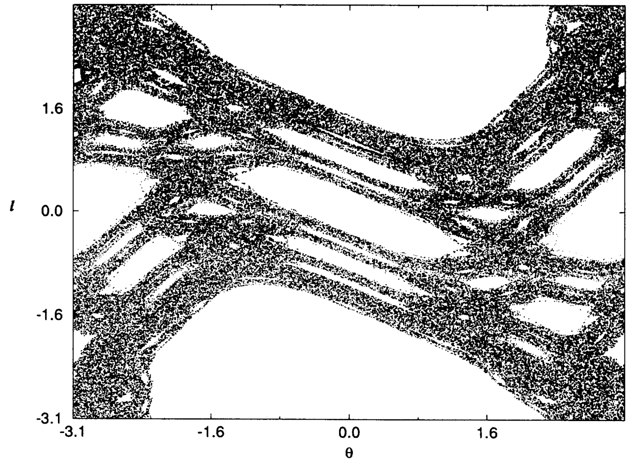

FIG. 2. Chaotic orbit for the classical map (7), with $V(\theta)=\cos(\theta)$ and $k=1.1$, corresponding to global chaos regime.

The above considerations apply as well to all systems which satisfy the condition (13), i.e., systems for which all the energy levels are integer multiples of some fixed energy, e.g., harmonic oscillator, a particle in a finite box or a spherical rotator. This phenomenon of quantum resonance is explained in view of the appearance of a new (one) time scale as a consequence of the quantization. For the harmonic oscillator, for which the classical system also admits one time scale, i.e., $1/\omega$, our results are valid classically as well, i.e., a two-sided kicked harmonic oscillator is not chaotic (even classically) under the condition (14). We therefore see that the suppression of chaos is strongly related to the existence of one time scale in the system, which in most cases is introduced due to the quantization process.

This quantum resonance effect is completely different from the quantum resonance phenomena introduced in Refs. [2,3]. In our model the resonance implies periodicity and therefore corresponds to strong localization of the system (in any representation), while for the mapping of Ref. [3] the resonance implies the existence of extended states (in time) and therefore ballistic motion [2].

We now consider the case for which the condition (14) is not satisfied, i.e., there is no resonance. In this case exact periodicity does not follow immediately from (12); however, one may analyze this expression using statistical methods in order to find the behavior of an ensemble of such systems. For such an ensemble, the wave function itself may be considered as a random variable, and one should study the moments of this random variable in order to obtain information on the behavior of the ensemble. We find that the distribution of $|\psi(\theta)|^2$ is exponential (Rayleigh statistics) provided that the dimensionless parameter $\gamma=(TE_1/4\pi\hbar)(\mathrm{mod}\,1)$ is of order one.

We reformulate (12) in order to make contact with the statistical theory of the diffusion (ladder) approximation in weakly disordered systems [7]. Using the relation $E_n=\hbar^2 n^2/2I$, one obtains

$$\psi(t+T)=\sum_{m,n} e^{-i\gamma(m^2+n^2)}\alpha_{mn}\eta_n \ , \tag{15}$$

where $\eta_n=\beta_n\psi_n$. It what follows we consider, for simplicity, the time evolution of energy eigenstates of the Hamiltonian H_0. It then follows that

$$I(\theta)\equiv|\psi(\theta,t+T)|^2=\sum_{m,n} e^{i\gamma(m^2-n^2)}\eta_n^*(\theta)\eta_m(\theta) \ . \tag{16}$$

We now study the statistical properties of an ensemble of quantum two-sided kicked rotators, which differ from each other by γ (e.g., different I or T). Following Ref. [3], we make the assumption that the phases $e^{-i\gamma(m^2+n_2)}$ are sufficiently random, and therefore may be treated as random numbers. Under this assumption, the only nonzero contributions for the statistical average of (16) arise for pairs for which the phase vanishes, e.g., $m=\pm n$. This implies that

$$\langle I\rangle=\sum_m|\eta_m|^2 \ . \tag{17}$$

It is easy to see [7] that after averaging over a sufficiently large set of initial conditions under which one can neglect the non-positive-definite terms, higher moments of I satisfy, according to the above argument, the Rayleigh statistics (exponential distribution)

$$\langle I^n\rangle=n!\langle I\rangle^n \ , \tag{18}$$

such that

$$P(I)=\frac{1}{\langle I\rangle}\exp\left[-\frac{I}{\langle I\rangle}\right] . \tag{19}$$

It can be shown that these arguments apply as well to the form of the wave function after n steps and therefore this statistical property holds generally.

In summary, we have shown that the quantization of the two-sided kicked rotator problem suppresses all the features of classical chaos, including the sensitive dependence on initial conditions, due to the existence of a new, single, time scale. This result indicates that one possible origin of the phenomenon of the suppression of chaotic effects in quantum systems is the introduction of *new* time scales, which does not exist classically in the problem. These considerations apply also to systems for which, although there exist several times scales, one time scale is the most relevant for the system. For example, in the Morse oscillator [8] the spectrum consists of two parts, namely, discrete and continuous spectrum, where the discrete part obeys the relation (13) [8]. Hence, while the classical two-sided kicked Morse oscillator is known [6] to be chaotic, one should expect a significant reduction of the chaotic effects due to quantization [for suitable choice of parameters according to (14)], in the region for which only the bound states of the oscillator are relevant.

We gratefully acknowledge I. Dana and R. Berkovits for stimulating and most helpful discussions. We are especially thankful to I. Dana for a critical reading of the manuscript.

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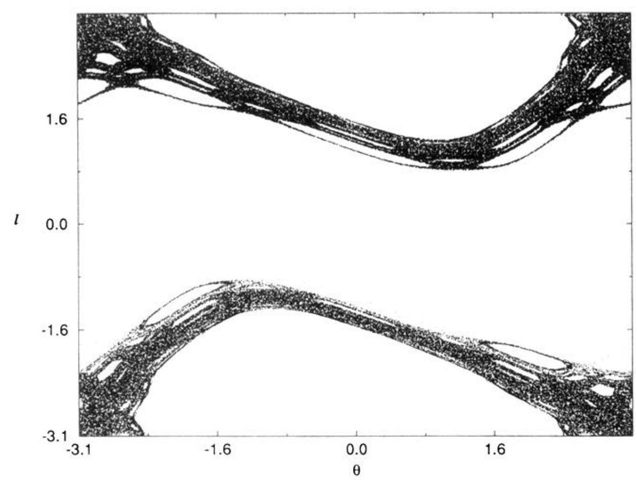

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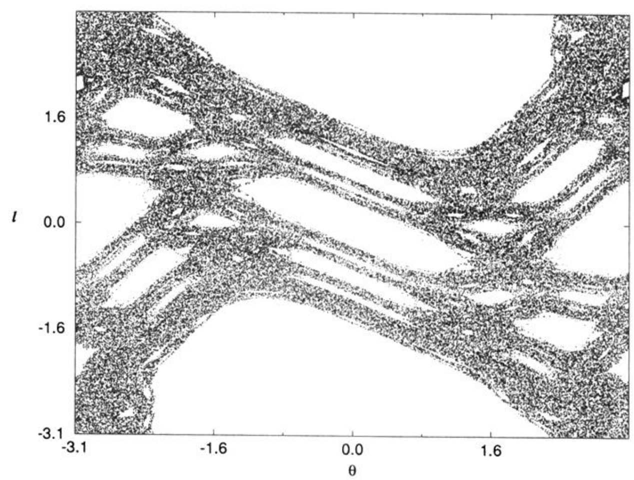

FIG. 2. Chaotic orbit for the classical map (7), with $V(\theta)=\cos(\theta)$ and $k=1.1$, corresponding to global chaos regime.