Quantization Process for the Driven Well—Where P Fails to Commute with P^2

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We consider the quantum mechanical behavior of a driven particle in an infinite 1D potential well. We show that the quantum dynamics of the system is induced by the delicate nontrivial properties of the momentum operator in this case, namely, its non-self-adjointness. Using this, we calculate the first order contribution to the cross section and the energy gain, and discuss their classical limit.

1. INTRODUCTION

In this paper we consider the quantum mechanical behavior of a driven particle in an infinite potential well. We discuss the proper way to apply a perturbative expansion to the system, and explain in detail the origin of the failure of the more naive approach. Our treatment applies also to strong fields in the high frequency limit. We now turn to a concise qualitative description of our approach.

Applying a gauge transformation, one can show that the Hamiltonian can be written as a function of the operators P and P^2 alone with time-dependent coefficients. Thus, it seems that the Hamiltonian commutes with itself for different times, and one may simply integrate over the time to get the evolution operator. The resulting evolution operator for a complete period (and a number of periods as well) behaves as one which follows from an integrable Hamiltonian. Moreover, for a properly chosen periods i.e., a proper choice of the start (and end) point, the complete period behavior is exactly the same as that of the unperturbed well.

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This result is physically unacceptable. It is well known[1,2] that the quantum dynamics of a system corresponds to its classical dynamics for time periods which scale (at least) like $\ln(1/h)$. This result can be rigorously proved using the path-integral formulation of the expansion around the WKB limit. Accordingly, since the classical driven particle experiences a chaotic motion, one should expect the quantum dynamics to show some features that resembles this chaotic (diffusive) behavior in the short time limit (although the long time limit is characterized by the energy-space localization which is a purely quantum mechanical effect). Nevertheless, if the quantum dynamics is determined by the integrated Hamiltonian, which is exactly equal to the one obtained from an integrable system, namely, the unperturbed well, it must also describe in this short time limit the classical dynamics related to this integrable Hamiltonian. However, the classical dynamics of these two are off course qualitatively different; for example, while the first diffuses over the whole stochastic layer, the second is energy-preserving. Note that the above-described derivation is exact for any value of h_{eff} (the effective h). Thus one may choose it to be arbitrarily small, such that the correspondence time diverges. One may therefore conclude that the result of this derivation contradicts the well-established "log-time" correspondence principle.

It turns out that the origin for this apparent contradiction is found in the delicate nontrivial properties of the P operator, when defined on a finite interval. We discuss some of the manifestations of this behavior, and explain in detail its implication on our derivation. It is shown that the seemingly trivial relation $[P, P^2] = 0$ does not hold for a particle in an infinite potential well, and therefore our gauged Hamiltonian *does not* commute with itself at different times. An explanation of this nontrivial behavior is given in terms of the self-adjoint extension procedure relevant to our model. This unexpected noncommutativity may be of importance to the analysis of many other finite quantum systems. In particular, each finite system under the influence of an electric or magnetic fields may be influenced by this property.

In order to study the effect of this noncommutativity of P and P^2 over the dynamics, we use the explicit form of the operator $[P, P^2]$ in the energy basis. We then employ the time-dependent perturbation theory to the interaction representation of the gauged Hamiltonian, which takes into account an infinite number of commutators $[P^2, [P^2, [..., [P^2, P]...]]]$, and obtain the first correction to the exact periodicity implied by the previous false derivation. Our expansion is appropriate even for cases in which $E_L = \lambda L$, the (maximal) potential difference between the ends of the well, is much larger then E_0, the spacing of the energy levels in the unperturbed well. The matrix elements of the perturbation are not small and, in

fact, the reason why the usual time-dependent perturbation theory does not break down is the approximate periodicity implied by the preceding wrong argument. We thus obtain an expression for the integral cross section, i.e., the probability to leave the initial state after one period of the perturbation. The high frequency limit of the frequency dependence of this cross section is $O(\omega^{-4})$, i.e., much faster then the usual $O(\omega^{-2})$ asymptotics, which characterizes the response in the high frequency limit.

The first-order treatment yields an infinite series for the escape probability—the probability *not* to return to the initial state after a complete period, and the energy gain. We discuss the asymptotics of these series in the various regimes. In particular, we study the behavior in the classical limit, i.e., $h_{eff} \to 0$, while keeping the energy fixed. We show how the approximate periodicity breaks down in this limit, thus furnishing the transition to the classically chaotic behavior. The classical limit of the energy gain is then compared to the corresponding classical quantity.

The rest of this paper is organized as follows: In Sect. 2 we present the naive "exact" solution of the model in terms of the transverse gauge. The surprising result is that the effects of the ac field are cancelled out over a complete period. We discuss the implications of this result. In Sect. 3 we explain the origin of the error in this derivation in terms of the algebraic properties of the operators involved with respect to their self-adjoint extension. Section 4 presents a systematic perturbative expansion around the limit in which the naive argument holds. This derivation is based on the interaction representation of the P operator. We apply time-dependent perturbative methods to calculate the first correction to the apparent exact periodicity. Asymptotic analysis of these perturbative results are discussed, with particular emphasis on the classical limit. Numerical results are presented, which verify the above perturbative derivation and test its range of applicability. Conclusions are given in Sect. 5.

2. FIRST CONSIDERATIONS

The Hamiltonian considered is

$$H = \frac{\bar{P}^2}{2m} + eEf(\omega t)\,\bar{X} \tag{1}$$

where $f(t)$ is a periodic function whose period is $T = 2\pi/\omega$. The wave-function is imposed to satisfy the boundary conditions of vanishing of the wave-function at the ends of the well $\bar{x} = 0$; $\bar{x} = L$. This model has been

investigated classically for the case $f(t) = \cos(\omega t)$,[3, 4, 6] and has been found to be chaotic. Using the dimensionless form, i.e.,

$$X = \bar{X}/L, \qquad \tau = t \tag{2}$$

it can be shown that the classical behavior of this model depends on a single parameter $\beta = eE/(m\omega^2 L)$. However, the quantum evolution depends also on an additional parameter the effective $\hbar$, $\hbar_{\text{eff}} = \hbar/(m\omega L^2)$. Using the dimensionless form, the Hamiltonian reads

$$H = \frac{P^2}{2} + \beta f(\tau)\, X \tag{3}$$

where the position representation of the scaled momentum operator is $P = -i\hbar_{\text{eff}}(d/dx)$. We now transform to the transverse gauge, in which the Hamiltonian takes the form

$$H = \tfrac{1}{2}(P - \beta F(\tau))^2 \tag{4}$$

with

$$F(\tau) = \int_0^\tau dt f(t) \tag{5}$$

The lower limit of the integral is $t = 0$ since we choose the field to be turned on at $t = 0$, and thus the vector potential should vanish at $t = 0$. At this point, one may naively use the following argument: since the Hamiltonian (4) commutes with itself at different times, the Schrödinger equation may be trivially integrated over τ, yielding the evolution operator $U(0, \tau) = \exp(-iH_{\text{eff}}(\tau))$ where

$$H_{\text{eff}} = \int_0^\tau H(t)\, dt = \frac{P^2 t}{2} - \beta P \int_0^\tau F(t)\, dt + \text{const} \tag{6}$$

The diagonalization problem for this effective Hamiltonian is rather trivial since for each τ it is just the Hamiltonian of a particle in an infinite well with a gauge term, whose eigenfunctions are of the form $\sin(kx)\exp(i\gamma(\tau)\,x)$.

A basic tool for analyzing the chaotic features of classical systems is the Poincaré section.[7, 8] For a time-periodic Hamiltonian system with one degree of freedom, the Poincaré surface of the section is just a strobe plot, that is, one plots (x, p) once every period τ. In particular, if the evolution of a system is periodic, the strobe plot will give a one-dimensional curve. On the other hand, if the evolution is chaotic, the trajectory will spread

over a two-dimensional region. Accordingly, most of the research interest
in the field of quantum chaos was directed to the study of the Floquet
operator, which determines the quantum time evolution of periodic systems
for times which are integer in the driving force period units.

Apparently, one sees that for our model, namely, the driven one-
dimensional potential well, the explicit form of the Floquet operator can be
easily obtained from Eq. (6) to be

$$F = H_{\text{eff}}(\tau = 2\pi) = P^2\pi + \beta P \int_0^{2\pi} dt F(t) + \text{const} \tag{7}$$

One obtains that the stroboscopic behavior of the system is rather trivial,
i.e., equivalent to that of a particle in an infinite well subjected to a con-
stant electric field. Moreover, for an appropriate choice of the function
$f(\omega t)$, e.g., $f(\tau) = \cos(\tau)$, the gauge term vanishes, and the Floquet operator
has no interaction dependence.

The above result is completely unacceptable physically, since it con-
tradicts the correspondence principle for arbitrary short times (with respect
to h). In particular although the classical particle in the well reaches (in
general) a completely different point in phase space after one period of the
perturbation, we have just shown that the quantum particle develops into
the state it should have been developed in the absence of the perturbation.
Thus, according to the correspondence principle, for $h \to 0$ the quantum
mechanical state should correspond to *two different classical* states, i.e.,
those points in phase space to which the system arrives in the presence and
in the absence of the driving force. This ambiguity manifests the apparent
breakdown of the correspondence principle (at least) for times of order of
the period, no matter how small is h. We therefore turn now to discuss the
origin of the failure of our derivation.

3. ALGEBRAIC PROPERTIES OF THE P OPERATOR

In order to find the problematic point in the above arguments. we
state explicitly the assumptions made above. Apart from simple algebra, we
used two basic relations i.e.,

$$[X, P] = ih_{\text{eff}}, \qquad [P, P^2] = 0 \tag{8}$$

The first assumption was used to perform the gauge transformation, and
the second in the statement that the gauged Hamiltonian (4) commutes
with itself for different τ's.

It seems that these two relations are well established. The first is the basic principle of quantum mechanics, whose failure leads to the breakdown of the whole method of canonical quantization in the system. The second assumption is just a manifestation of the associativity of operator multiplication. However, this second assumption turns out to be false, as we now explain in detail.

The simplest way to realize the failure of the relation $[P, P^2] = 0$ is through the matrix representation of these operators in the energy basis of the unperturbed well. The matrix elements of these matrices are given by

$$(P^2)_{mn} = \hbar_{\text{eff}}^2 \pi^2 m^2 \delta_{mn} \tag{9}$$

$$P_{mn} = \begin{cases} -i\hbar_{\text{eff}} \dfrac{4mn}{m^2 - n^2}, & m + n \text{ odd} \\ 0, & m + n \text{ even} \end{cases} \tag{10}$$

One clearly sees that these matrices do not commute. In fact the commutator gives

$$([P^2, P])_{mn} = -4i\hbar_{\text{eff}}^3 \pi^2 mn \tag{11}$$

In terms of this matrix representation, the reason for the unassociativity of the matrix multiplication is that this multiplication involves summations which do not absolutely converge, and thus the sum depends on the order of summation. However, since one expects operators to be associative, the question still holds whether this strange result really manifests an operator property (and if so, why?), or is only an artifact of the particular basis used.

First, we note that the P operator possesses some delicate properties when restricted to a finite interval. Physically, due to the uncertainty principle and the finite size of the well, an exact measurement of the momentum is impossible, and thus there is no eigenstate of P in the Hilbert space, as can be seen also by applying the appropriate boundary conditions to the differential eigenvalue problem. Moreover, it can be easily seen that since P is canonically conjugate to X, it generates spatial translations. Thus its domain is restricted to functions which do not leave the well for infinitesimal translations. A well-known criterion for the commutativity of two operators is the existence of a basis in which the two of them are diagonal. As we have seen, P has an empty spectrum and thus cannot be diagonalized in any basis. In particular it is not diagonal in the basis in which P^2 is. Thus they do not commute.

The reason for this unassociativity is as follows. The operator P^2 in the Hamiltonian is not really defined as $P \cdot P$ but rather as the self-adjoint

extension of this form. P itself is not an essentially self-adjoint operator when restricted to a finite interval,[9] and has therefore no self-adjoint extension.[10] Thus, although the "bare" differential operators P and P^2 (i.e., the operators defined only on the (twice) differentiable functions whose derivative vanishes on the boundaries, without the self-adjoint extension) commute on their common dense domain, after the procedure of self-adjoint extension, the domain of P^2 is much larger than that of P. If there had been such an extension for the P operator, its square would have been the extension of the P^2 operator (uniqueness of the extension). However, since due to the above (both mathematical and physical) reasons, P cannot be extended to a self-adjoint form, the "bare" P does not commute with the extended P^2.

In the next section we study the perturbative expansion with respect to the terms obtained due to the uncommutativity.

4. PERTURBATIVE APPROACH

The starting point for the perturbative expansion is the dimensionless form of the gauged Hamiltonian

$$H = \frac{(h_{\text{eff}} P - \beta F(\tau))^2}{2} = h_{\text{eff}}^2 \frac{P^2}{2} - 2h_{\text{eff}} \beta F(\tau) P + \text{const} \tag{12}$$

where P here is the dimensionless momentum operator, whose matrix elements in the energy representation are

$$(P^2)_{mn} = h_{\text{eff}}^2 m^2 \pi^2 \delta_{mn} \tag{13}$$

$$P_{mn} = \begin{cases} -ih_{\text{eff}} \dfrac{4mn}{m^2 - n^2}, & m + n \text{ odd} \\ 0, & m + n \text{ even} \end{cases} \tag{14}$$

We now take into account the fact that P and P^2 do not commute, and treat the P term as a time-dependent perturbation to the Hamiltonian $H_0 = P^2/2$. We use the interaction picture in which

$$P_{mn}^I(\tau) = (\exp^{i\tau H_0/h_{\text{eff}}} P \exp - i\tau H_0/h_{\text{eff}})_{mn}$$

$$= \begin{cases} -ih_{\text{eff}} \dfrac{4mn}{m^2 - n^2} \exp(i\alpha\tau(m^2 - n^2)), & m + n \text{ odd} \\ 0, & m + n \text{ even} \end{cases} \tag{15}$$

where $\alpha = \hbar_{eff}\pi^2/2$. The latter equation can be derived simply by taking the matrix element of the whole expression in the middle of the above equation, and applying the exponential operators to the bra and ket states (these are eigenstates of the exponential operators). However, another way to look at this equation is as a sum over the commutator series as follows:

$$(\exp(i\tau H_0/\hbar_{eff})\; P\; \exp(-i\tau H_0/\hbar_{eff}))_{mn}$$

$$= \sum_{n=0}^{\infty} \frac{(i\tau/2)^n}{n!} [P^2, P^2,...[P^2, P]...]_{mn}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{mn}{m^2 - n^2} (i\tau\alpha)^n (m^2 - n^2)^n$$

$$= \frac{mn}{m^2 - n^2} \exp(i\alpha\tau(m^2 - n^2)) \tag{16}$$

It is therefore clear that in the perturbative approach one obtains an infinite number of commutators in the derivation, and thus the problems described in the previous sections are avoided.

We demonstrate the approach for the calculation of the escape probability, i.e., the probability to leave the initial state m after one period of the perturbation, to second order in β. The amplitude for a transition to the n state is given by

$$A(m \rightarrow n) = \beta P_{mn} \int_0^{2\pi} d\tau \sin(\tau) \exp(i\alpha\tau(m^2 - n^2)) \tag{17}$$

The escape probability (integral cross section) is therefore given by

$$\sigma = \sum_n |A(m \rightarrow n)|^2$$

$$= 32\beta^2 m^2 \sum_{m+n\,\text{odd}} \frac{n^2}{(m^2 - n^2)^2} \frac{1 - \cos(2\pi\alpha(m^2 - n^2))}{(\alpha^2(m^2 - n^2)^2 - 1)^2} \tag{18}$$

Figure 1 presents a comparison of this first-order result for the escape probability and the energy gain after one period, with a full numerical solution of the Schrödinger equation obtained through a quality control Runge–Kutta method. The agreement in both cases is quite good.

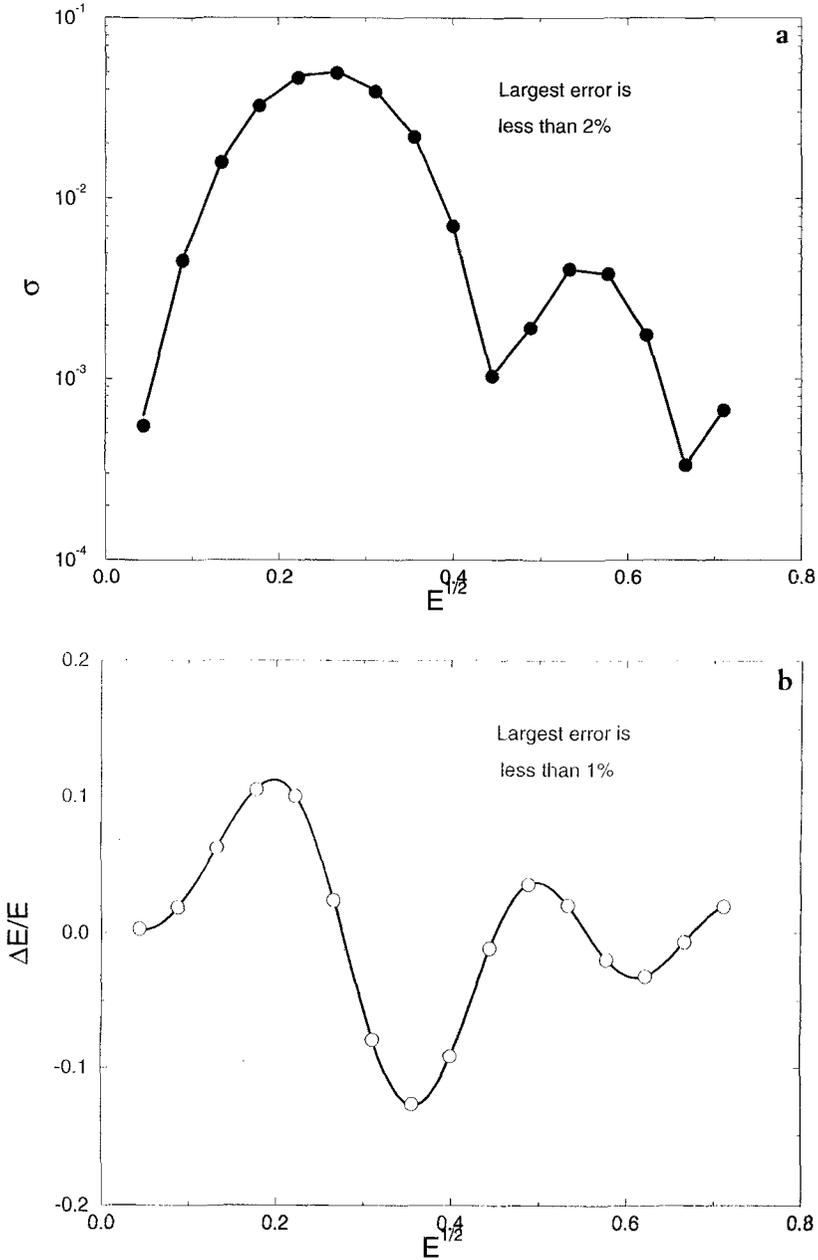In the following, we estimate this sum for the various relevant regimes.

Fig. 1. A comparison of the numerical summation of the first-order expression (full lines) with the numerical integration of the Schrödinger equation (symbols). $\hbar_{\text{eff}} = 0.02$ and $\beta = 0.005$. (a) The escape probability; (b) the energy gain.

4.1. Low-Energy Regime

We first consider the regime in which the sum is appropriately estimated by taking the continuous limit (this regime will be defined precisely later), i.e. replacing the sum by the integral

$$\frac{\sqrt{\alpha}}{2} I = \frac{\sqrt{\alpha}}{2} \int_0^\infty f(x) \, dx = \frac{\sqrt{\alpha}}{2} \int_0^\infty \frac{x^2}{(z-x^2)^2} \frac{1 - \cos(2\pi(z - x^2))}{((x^2 - z)^2 - 1)^2} \, dx \quad (19)$$

where $z = \alpha m^2$, and x was defined through $x = \sqrt{\alpha} \, n$. We now consider two different subregimes. For $z \ll 1$, one may estimate the integral by its value for $z = 0$, which is $I = 4.663...$. The condition for replacing the sum by the integral is that the spacing of the summation points, i.e., $2\sqrt{\alpha}$, is sufficiently small compared to the scale of changes in the function $f(x)$. In this subregime this condition is equivalent to $\sqrt{\alpha} \ll 1$. This, together with the condition $z \ll 1$, implies $E = 2z\alpha/\pi^2 \ll 1$. Thus the regime in which the integral approach is appropriate is the low-energy regime. However, where $z \ll 1$, the condition $\sqrt{\alpha} = \sqrt{z}/m \ll 1$ holds trivially since m, the quantum number, is a positive integer, $m \geqslant 1$. Thus the limit $z \ll 1$ is relevant only to the low-energy regime.

For the $z \gg 1$, one should notice that the integrand in (19) is a highly peaked function around the region $x^2 = z \pm 1$ (see Fig. 2). The maximum of the function is obtained at $x^2 = z$ and its value is $2\pi^2 z$, while for $x^2 = z \pm 1$ the integrand is $(\pi^2/2)(z + 1)$, i.e., about a one-fourth of the maximum. The decrease of the function away from this region is $O((x^2 - z)^{-6})$, and thus one may neglect the contributions to the integral from any region other than the maximum. The width of the peak is therefore approximated by the distance between the two points $x^2 = z \pm 1$ i.e., $\Delta = 1/\sqrt{z}$. Thus one obtains for the $z \gg 1$ regime the estimate $I \sim 2\pi^2 \sqrt{z}$. Numerical integration shows that while the z dependence of the integral is as stated above, a more accurate value of the coefficient is

$$I = 14.8044 \sqrt{z} \quad (20)$$

The condition for the validity of the integral approximation in this case is that the spacing of the sum is less than the peak width, i.e., $2\sqrt{\alpha} \ll 1/\sqrt{z}$, which implies $E = 2z\alpha/\pi^2 \ll 1$.

Accordingly, the following results for the escape probabilities in the low-energy limit are obtained:

$$\sigma = \begin{cases} 6.8058\beta^2 E/\hbar_{\text{eff}}^{3/2}, & z \ll 1 \\ 9.7268\beta^2 E^{3/2}/\hbar_{\text{eff}}^2, & z \gg 1 \end{cases} \quad (21)$$
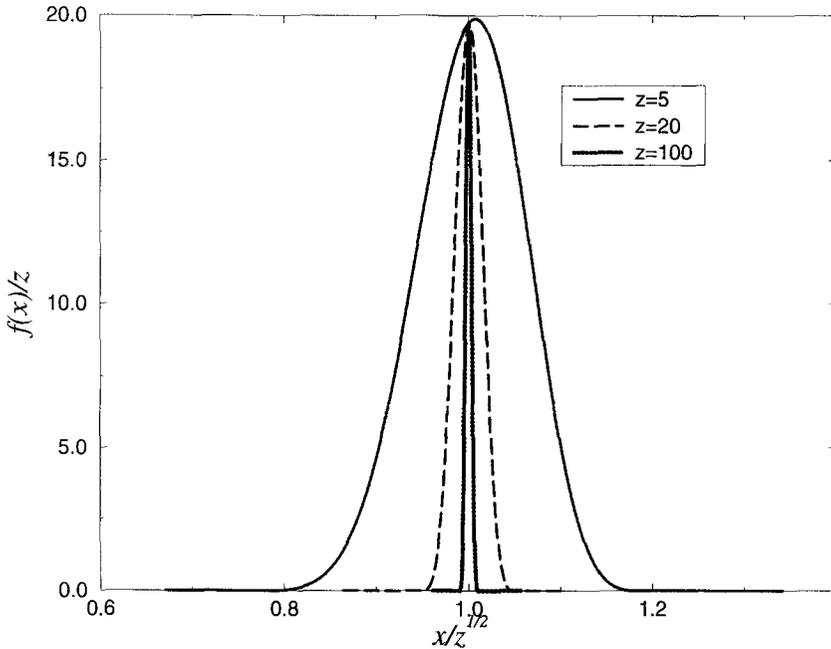
Fig. 2.   The function $f(x)$, the integrand used for the integral approximation, for various values of z. The function is scaled by a factor $1/z$.

In Fig. 3 we show the results of a numerical summation of the 1st-order series, compared to the two asymptotes. The transition between the two regimes is clearly seen.

The nature of the transition from the extreme quantum regime where the system is almost periodic (as implied by the wrong argument of Sec. 2) to the classical regime where the dynamics turns out to be chaotic over larger and larger time scales can now be clearly seen. As one approaches the classical limit $\hbar_{\mathrm{eff}} \to 0$, the approximate periodicity of the system is lost, since the escape probability, i.e., the deviation from periodic behavior, diverges as $\hbar_{\mathrm{eff}}$ approaches zero. One may also define the time scale after which the periodicity is ruined as the inverse of the escape probability. i.e.,

$$\tau_{\mathrm{chaotic}} \sim T/\sigma \tag{22}$$

and this time scale vanishes in the classical limit. It is interesting to note that while the existence of an upper limit (in time) for the correspondence principle to hold is well established and discussed in recent years,[1, 2] the system under consideration exhibits the opposite type of behavior, i.e., a lower bound on the time in which correspondence holds.
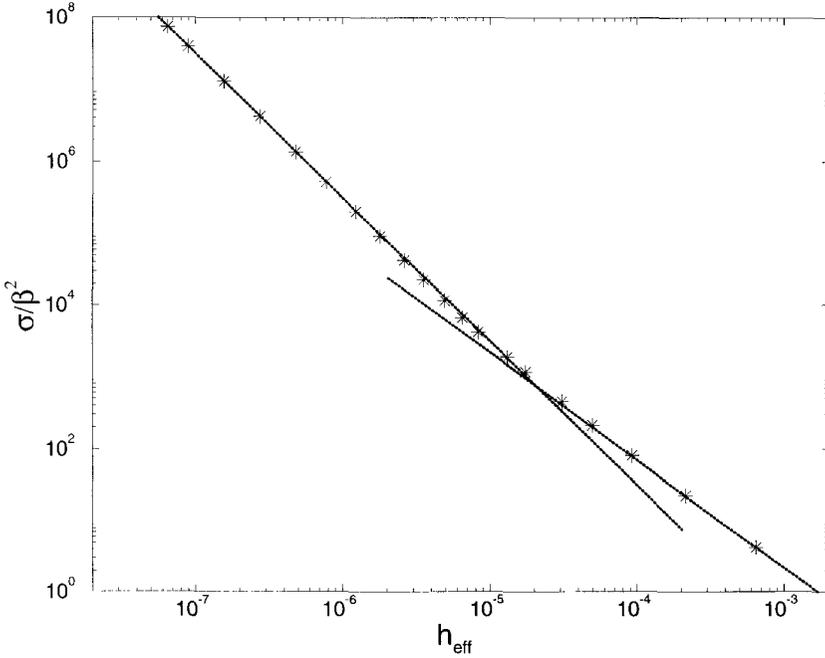
Fig. 3.   Exact summation of the first-order expression for the escape probability, compared to the two asymptotes. The stars correspond to the exact summation results and the lines to the asymptotes. The (dimensionless) energy is $5 \cdot 10^{-5}$.

We now apply similar arguments to the calculation of the energy change of the driven particle after one period. This quantity has a concrete classical meaning and thus this example may further clarify the nature of the classical limit. Moreover, the energy absorption of the quantum well is also an experimentally measurable quantity, and thus one may study experimentally the complete transition from the extreme quantum regime to the classical limit, by changing the parameters of the well such that h_{eff} is in the appropriate regime. The arguments concerning the integral estimations are quite similar to those described above, and the results are

$$\Delta E = \begin{cases} 20.426\beta^2 E/\sqrt{h_{\mathrm{eff}}}, & z \ll 1 \\ 17.772\beta^2\sqrt{E}, & z \gg 1 \end{cases} \tag{23}$$

Figure 4 shows the energy gain per period as a function of h_{eff}, obtained from a numerical summation of the appropriate 1st-order sum. The two asymptotes are also indicated on the graph, and the agreement seems quite good.
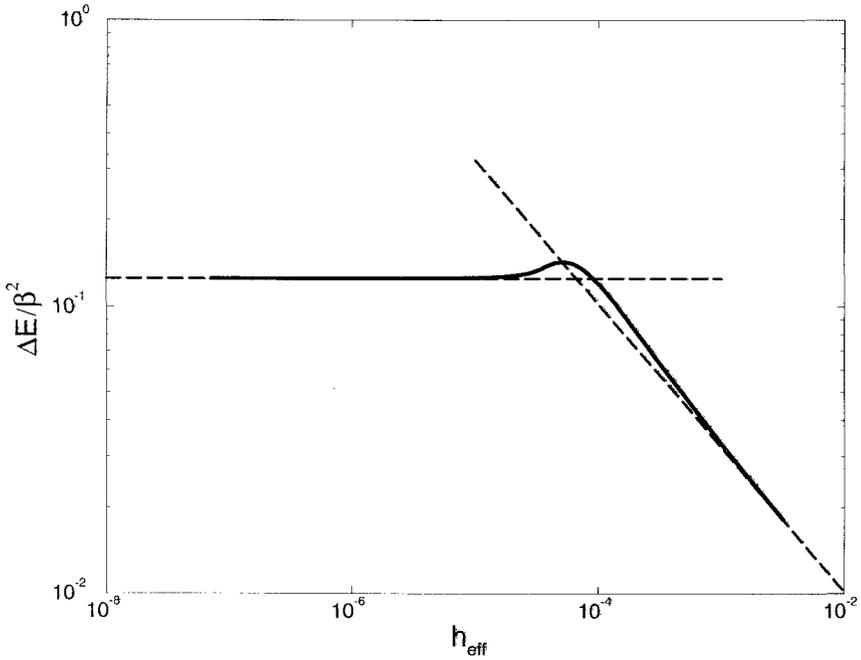
Fig. 4. Same as the previous figure for the energy gain. The thick line corresponds to exact summation results and the straight thin lines to the asymptotes.

One can clearly see that for a fixed energy E, as $h_{\text{eff}} \to 0$, $z = E/h_{\text{eff}}$ approaches infinity. Thus the relevant regime to the classical limit is the second one, in which the quantity converges to the well-defined limit ΔE (classical) $= 17.772324\beta^2 \sqrt{E}$. In Fig. 5 Monte Carlo calculation for the local diffusion constant as a function of β is presented. The functional form is indeed $O(\beta^2)$ as implied by the quantum calculation, and the value of the coefficient (scaled by $\sqrt{E}$) is $17.77 \pm .04$.

We wish to note here one more point. Considering the high-frequency limit of this problem, it is clearly seen that while $\omega \gg 1$, both h_{eff} and β approach zero. Thus, the pertubative, classical-limit calculation holds and one may conclude that the high-frequency asymptote of the absorption is $O(\omega^{-4})$ rather than the usual $O(\omega^{-2})$. This is another manifestation of the approximate periodicity of the system in the extreme quantum regime.

4.2. High-Energy Regime

As we have shown above, in the high-energy regime only the limit $z \gg 1$ should be considered. We can then maintain the picture of a highly
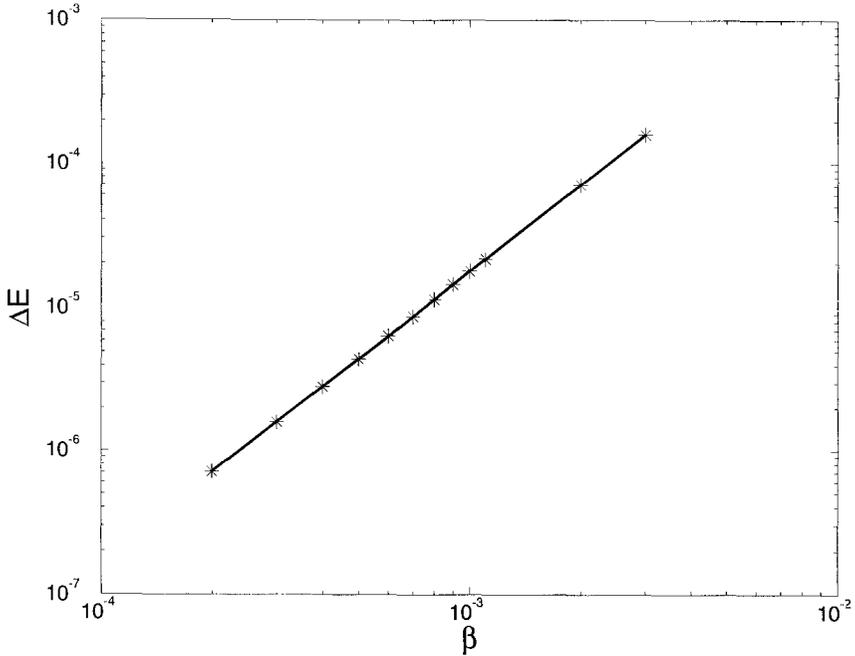
Fig. 5. The energy gain after one period, as obtained from averaging over $8 \cdot 10^7$ classical
paths at energy $E = 5 \cdot 10^{-5}$. The best fitted coefficient is 17.77 ± 0.04.

peaked function $f(x)$ as in the low-energy case. However, here the function
values are taken in large steps compared to the peak width. Since the decay
of $f(x)$ away from the peak is extremely fast, we can approximate the sum
by the contribution of the closest points, i.e., $n = m \pm 1$. This implies (for
simplicity we assume that the initial state m is not close to the ground
state, $m \gg 1$)

$$\sigma = \frac{32\beta^2 E}{\hbar_{\mathrm{eff}}^2 \pi^2 (2\pi^2 E - 1)^2} (1 - \cos(\pi^2 \sqrt{8E})) \qquad (24)$$

The E dependence of σ ($\hbar_{\mathrm{eff}}$ fixed) is presented in Fig. 6, as obtained from
summation of the first-order series. The transition between the three dif-
ferent regimes is clearly seen. For low energies, $z \ll 1$ and the behavior is
$O(E)$. In the intermediate regime E is still small, but z is large and one
obtains an $O(E^{3/2})$ dependence [See Eq. (21)]. For $E > 1$ the oscillatory
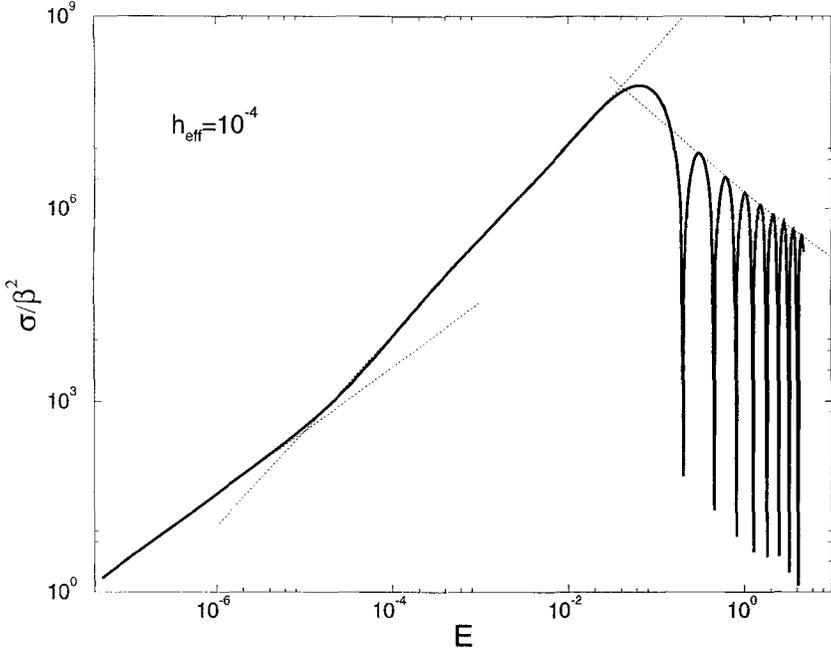behavior of Eq. (24) dominates. The peaks height decays like E^{-1}, as

Fig. 6.   The escape probability σ/β^2 as a function of E. The effective h value is 0.02. The right dotted line corresponds to the maximum values in the high-energy regime, as obtained through the closest-points approximation. The other lines correspond to the two low-energy asymptotes.

implied by Eq. (24). The straight dotted line corresponds to the function

$$\frac{64\beta^2 E}{h_{eff}^2 \pi^2 (2\pi^2 E - 1)^2}$$

and passes through the maxima obtained from Eq. (24).

A similar analysis is done for the energy absorption. Again, we calculate the classical limit of the quantum result and get

$$\Delta E = \frac{16E(2\sqrt{2E}\,\pi^2 \sin 2\pi^2 \sqrt{2E}\,(2\pi^2 E - 1) - (2\pi^2 E + 3)(1 - \cos 2\pi^2 \sqrt{2E}))}{(2\pi^2 E - 1)^3}$$

$$(25)$$

This result is obtained using just the above analytical argument and much tedious but straightforward series expansions.

5. CONCLUSION

The ac-driven particle in an infinite potential well is considered. This system, which is classically chaotic presents a rich quantum behavior.

In the first sections we have shown that all the nontrivial properties of the system are related to the fact that in the infinite potential well the relation $[P, P^2] = 0$ does not hold. If it did, the evolution would become exactly periodic, which contradicts the chaotic behavior in the classical limit. It is worth noting that here the *classical* limit depends upon some *noncommutativity* property, which seems to be a pure quantum-mechanical effect.

Apart from being crucial for the analysis of our system, the noncommutativity of P and P^2 may turn out to be of importance in many cases in which the response of quantum wells and other finite quantum systems to electric and magnetic fields is studied.

In the second part of the paper, a perturbative expansion around the limit in which the system is periodic was developed. The various regimes were considered, and the analytical estimates for the different asymptotes were obtained. These were tested with respect to an exact summation. The classical limit was also discussed. It was shown how the approximate periodicity breaks down as $\hbar_{\text{eff}} \to 0$. A classical limit for the absorption was obtained from the classical limit of the quantum calculations, and it was shown that it agrees with the classical result obtained from integrating the classical equations of motion.

Finally we note that this system, including all the various regimes above, is experimentally realizable, using quantum wells radiated by a laser. Different values of the width of the well and the laser frequency control the parameters $\hbar_{\text{eff}}$ and β such that the whole parameter space is accessible. The influence of perturbations caused by impurities and imperfections in the well, dissipation due to phonon excitation, finiteness or distortion of the well, motion in the plane of the well, and even electron–electron interactions were discussed in Ref. 11 and subsequent works, and it is claimed that the one-particle, 1D, ideal system effects can in fact be realized.

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9. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, (Academic, New York, 1975), Vol. II, Chap. X.
10. In fact, as is clearly demonstrated in Ref. 9, during the mathematical procedure of extending the differentiation operator defined on functions that vanish on both sides of the interval, one must release these too restrictive boundary conditions, and consider all periodic functions $\psi(0) = \psi(L)$. This changes the whole physical situation. Thus, in fact, mathematics tells us that if we want to have a self-adjoint momentum operator, we should consider a *ring* and not a *well*.
11. B. Galdrikian, B. Birnir, and M. Sherwin, *Phys. Lett. A* **203**, 319 (1995).