Antiresonance and localization in quantum dynamics

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The phenomenon of quantum antiresonance (QAR), i.e., exactly periodic recurrences in quantum dynamics, is studied in a large class of nonintegrable systems, the modulated kicked rotors (MKRs). It is shown that asymptotic exponential localization generally occurs for η (a scaled $\hbar$) in the infinitesimal vicinity of QAR points η_0. The localization length ξ_0 is determined from the analytical properties of the kicking potential. This ''QAR localization'' is associated in some cases with an integrable limit of the corresponding classical systems. The MKR dynamical problem is mapped into pseudorandom tight-binding models, exhibiting dynamical localization (DL). By considering exactly solvable cases, numerical evidence is given that QAR localization is an excellent approximation to DL sufficiently close to QAR. The transition from QAR localization to DL in a semiclassical strong-chaos regime, as η is varied, is studied. It is shown that this transition takes place via a gradual reduction of the influence of the analyticity of the potential on the analyticity of the eigenstates, as the level of chaos is increased. [S1063-651X(96)02711-0]

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I. INTRODUCTION

The study of ''quantum chaos,'' i.e., understanding the ''fingerprints'' of classical chaos in quantum mechanics [1,2], has led to the discovery of a variety of new quantum-dynamical phenomena. Several such phenomena occur in time-periodic systems described by the general Hamiltonian

$$H=H_0+H_1 f(t), \tag{1}$$

where H_0 is some time-independent Hamiltonian, H_1 is a perturbation, and $f(t)$ is periodic with period T, $f(t+T)=f(t)$. In many cases, $f(t)$ is chosen, for simplicity, as a periodic delta function $f(t)=\Delta_T(t)\equiv\Sigma_{s=-\infty}^{\infty}\delta(t-sT)$, giving the well-known class of ''kicked'' systems. Representative models in this class are the kicked rotor [3–12], the kicked Harper model [13], and the kicked harmonic oscillator [14].

The quantum dynamics of time-periodic systems (1) is governed by their quasienergy (QE) spectrum (i.e., the spectrum of the one-period evolution operator). Different properties of the QE spectrum lead to quantum-dynamical phenomena having, in general, no classical analog. A classic example is the quantum suppression of chaotic diffusion in the kicked rotor (KR) [3], accompanied by quasiperiodic recurrences [5]. An important interpretation of this phenomenon has been given [6,7] by showing first that, in the angular-momentum representation, the QE eigenstates of the KR satisfy the equation describing a one-dimensional (1D) tight-binding model with pseudorandom disorder. This pseudorandomness is generic, as it exists for almost all (irrational) values of a scaled (dimensionless) $\hbar$, which we denote here by η. It was found [15] that in several interesting cases pseudorandom tight-binding models exhibit localization properties similar to those of truly random ones (Anderson localization) [16]. Assuming the general occurrence of this localization, it follows that, generically, the QE eigenstates are exponentially localized in angular momentum and the QE spectrum is pure point. This localization in pseudorandom tight-binding models, equivalent to quantum-dynamical systems with nonintegrable classical counterparts, is called *dynamical localization* (DL) [17]. The quantum suppression of diffusion in the KR is an immediate consequence of DL. Despite the fact that DL has no classical analog, there exists a remarkable and simple relation between the classical chaotic-diffusion coefficient D in the KR and the *asymptotic* DL length ξ in the semiclassical regime (sufficiently small η): $\xi\approx D/2$ [3,7–9]. For nongeneric, rational values of η, in all the kicked systems the phenomenon of quantum resonance occurs [4], i.e., the quadratic increase of the energy expectation value with time. This phenomenon is due to an absolutely continuous QE spectrum, exhibiting a band structure.

In this paper, DL is approached in the light of a different kind of phenomenon for systems (1): exactly *periodic* recurrences. This phenomenon is defined, in general, by

$$U^p=e^{-i\beta}, \tag{2}$$

where U is the one-period evolution operator for (1), $e^{-i\beta}$ is some constant phase factor (a c number), and p is the smallest positive integer for which (2) is satisfied. Thus pT is the recurrence period. As it will become apparent in this paper, the phenomenon (2) may occur, in general, only for very special values of η and it is thus nongeneric. In fact, for the general class of systems introduced in this paper, it occurs precisely at values of $\eta=\eta_0$ corresponding to quantum resonances [4] in the kicked systems. At the same time, this phenomenon, manifesting itself in bounded, periodic variation of expectation values, is diametrically opposite to quantum resonance. We shall therefore refer to (2) as to the *quantum antiresonance* (QAR) phenomenon.

While this phenomenon is nongeneric, we show in this paper that, for a large class of nonintegrable systems, it is generally accompanied by a very interesting effect: In the immediate vicinity of QAR (infinitesimal $\eta-\eta_0$), *asymptotic exponential localization* with a pure-point QE spectrum takes place. The existence of this ''QAR localization'' is rigorously established in the framework of a self-consistent approach, which allows for an exact determination of the asymptotic localization length ξ_0. For $\eta_0=0$, the QAR localization is associated with an integrable limit of the classical Hamiltonian. On the other hand, values of $\eta_0\neq 0$ usually correspond to the strong quantum regime of a nonintegrable Hamiltonian, exhibiting chaotic diffusion.

As in the case of the KR and other systems [6–8,10,11], we show that our class of systems can be mapped into tight-binding models with pseudorandom disorder (for η sufficiently irrational). DL is then expected to occur generically in our systems. For η infinitesimally close to η_0, the tight-binding models are *not* defined, so that QAR localization *cannot* be viewed, strictly speaking, as a kind of DL. However, if η is sufficiently irrational and close to η_0, one expects DL to take place and to exhibit approximately the same features as those of QAR localization. In particular, the DL length ξ should be well approximated by the QAR-localization length ξ_0, and one expects that $\xi\to\xi_0$ as $\eta\to\eta_0$. We provide strong numerical evidence that this is indeed the case. Since ξ_0 can be determined exactly, this seems to be the first case where a DL length ξ can be found with arbitrary accuracy in nonintegrable systems.

We also study the dependence of ξ on $\eta-\eta_0$ for η not very close to η_0. This allows one to understand the transition from QAR localization to DL in regimes basically different in nature from QAR. Values of η sufficiently far from η_0 may correspond to semiclassical regimes of local or global chaos. As already mentioned, the semiclassical regime of global chaos in KR systems is characterized by the approximate relation $\xi\approx D/2$ [3,7–9]. Calculations of ξ for the KR were performed [7,8] by applying the method of minimal Lyapunov exponent [18] to an equivalent pseudorandom tight-binding model. The method is based on a finite truncation of the (generally infinite) vector of hopping constants, so that only the first N neighbors are kept. Then $\xi=\lim_{N\to\infty}1/\gamma_N$, where γ_N is the minimal Lyapunov exponent of a $2N$-dimensional symplectic map associated with the truncated model. We show that the truncated model has, for all N, a well-defined dynamical equivalent exhibiting QAR. Then, by studying numerically the dependence of γ_N on both N and η, we show that the transition from QAR localization to DL in a semiclassical strong-chaos regime takes place via a gradual reduction of the influence of the analyticity of the system on the analyticity of the eigenstates, as the level of chaos is increased.

In the simple case of $N=1$, i.e., a nearest-neighbor pseudorandom ''Lloyd model,'' we derive the *exact* relation $\xi_0=D/2$. Since many numerical calculations indicate that ξ is independent of η for such a model (see, e.g., Refs. [7,15]), this is strong evidence that the relation $\xi=D/2$ holds exactly for the corresponding dynamical system. A lengthy derivation of this relation was given in Ref. [9], based on the assumption that the pseudorandom disorder can be replaced by a truly random one (this gives the usual Lloyd model [19]).

The paper is organized as follows. In Sec. II, we discuss some basic aspects of the QE spectrum at QAR and consider special cases of QAR occurring in ordinary KR systems. In Sec. III, we introduce the general class of modulated kicked rotors (MKRs) and determine values of η,η_0, where QAR of period $p=1$ occurs for *arbitrary* kicking potentials (more general cases of QAR, of periods $p=1$ and $p=2$, are considered in Appendix A; in Appendix B, we study QAR of arbitrary period for integrable versions of the MKRs). We show, on the basis of a self-consistent approach, that for infinitesimal $\eta-\eta_0$ asymptotic exponential localization takes place, with a pure-point QE spectrum. This spectrum and the QE states are determined from an effective Hamiltonian with a periodic potential and the localization length ξ_0 is fixed by the analytical properties of this potential. If $\eta_0=0$, the effective Hamiltonian turns out to be precisely an integrable limit $(T\to 0)$ of the classical MKR Hamiltonian. In Sec. IV, we consider cases for which exact and closed results concerning QAR localization (e.g., ξ_0) and the associated QE spectrum can be obtained. Using these results, we provide strong numerical evidence that QAR localization is an excellent approximation to DL sufficiently close to QAR. In Sec. V, we show how MKR dynamical problems can be mapped, in general, into multichannel tight-binding models [20] with pseudorandom disorder. In Sec. VI, we study the transition from QAR localization to DL in a strong-chaos regime for a simple MKR system equivalent to the KR. This study is performed by considering the minimal Lyapunov exponents γ_N for successive truncations of the corresponding pseudorandom tight-binding model. The exact relation $\xi_0=D/2$ is derived for a nearest-neighbor pseudorandom Lloyd model. Conclusions are presented in Sec. VII. Some of our results have been briefly reported in Refs. [21,22].

II. QE SPECTRUM AT QAR: EXAMPLES OF QAR IN KR SYSTEMS

An immediate consequence of (2) is that the spectrum of U consists precisely of p eigenvalues $\exp(-i\omega_l)$, $l=0,\ldots,p-1$, where the quasienergies ω_l are given by

$$\omega_l=\frac{\beta+2\pi l}{p}. \tag{3}$$

Since the QE spectrum is finite, each quasienergy (3) must be infinitely degenerate. An infinite set of QE states associated with QE level l is obtained by applying the corresponding projection operator for the cyclic group $\{e^{is\beta/p}U^s\}_{s=0,\ldots,p-1}$ to all the states Ψ in the Hilbert space:

$$\psi_l=\frac{1}{p}\sum_{s=0}^{p-1} e^{is\omega_l}U^s\Psi. \tag{4}$$

We recall here that in the case of quantum resonance the QE spectrum consists of a finite number of bands [4]. The finite width of each of these bands leads to ballistic motion (quadratic increase of the energy expectation value with time). In the QAR case, on the other hand, one has the diametrically opposite phenomenon of periodic recurrences. This phenomenon has nothing to do with localized QE states, since the infinite basis of states (4) for QE level l can be chosen, of course, either localized or extended by properly choosing the state Ψ. The periodic recurrences may be explained by saying that each of the p infinitely degenerate levels in the QAR case is the extreme limit case of a quantum-resonance band of zero width. This point of view will become clearer by the following examples.

A first case of QAR was noticed by Izrailev and Shepelyanskii [4] in the KR. Consider the general KR Hamiltonian

$$H=\frac{L^2}{2I}+\hat{k}V(\theta)\Delta_T(t), \tag{5}$$

where L is the angular momentum, I is the moment of inertia, $\hat{k}$ is a parameter, and $V(\theta)$ is a general periodic function of the angle θ. The evolution operator for (5), from $t=-0$ to $t=T-0$, is

$$U=e^{-i\tau\hat{n}^2}e^{-ikV(\theta)}, \tag{6}$$

where $\hat{n}\equiv L/\hbar=-id/d\theta$, $\tau\equiv\hbar T/2I$, and $k\equiv\hat{k}/\hbar$. Quantum resonances occur, in general, for rational values of $\eta=\tau/2\pi$ [4]. Consider, however, the special case of $\eta=1/2$. Using the relation

$$e^{-i\pi\hat{n}^2}e^{-ikV(\theta)}=e^{-ikV(\theta+\pi)}e^{-i\pi\hat{n}^2}, \tag{7}$$

which is easily established by comparing the Fourier expansions of $V(\theta)$ and $V(\theta+\pi)$, one finds in this case that

$$U^2=\exp\{-ik[V(\theta)+V(\theta+\pi)]\}.$$

Thus the condition (2) for QAR is satisfied with $p=2$ if $V(\theta)+V(\theta+\pi)=\beta/k$ identically, for some β. This implies that $V(\theta)$ must have the general Fourier expansion

$$V(\theta)=\frac{\beta}{2k}+\sum_{s=-\infty}^{\infty} v_{2s+1}e^{i(2s+1)\theta}. \tag{8}$$

This is, of course, the case for the standard potential $V(\theta)=\cos(\theta)$, considered in Ref. [4]. According to (3), the QE spectrum consists of two infinitely degenerate levels, $\omega=\beta/2,\beta/2+\pi$. By ''switching on'' even-harmonic components v_{2s} in (8), the infinite degeneracy is removed and the two levels broaden into two bands, corresponding to the generic spectrum of the 1/2 quantum resonance.

More general results can be obtained for the ''linear'' version of (5) [23], which is, however, integrable [24]:

$$H=\frac{\tau}{T}L+\hat{k}V(\theta)\Delta_T(t), \tag{9}$$

where τ is now some dimensionless parameter. The evolution operator for (9) is

$$U=e^{-i\tau\hat{n}}e^{-ikV(\theta)}. \tag{10}$$

The pth power of U in (10) can be easily given in closed form

$$U^p=\exp\left[-ik\sum_{s=1}^{p} V(\theta-s\tau)\right]\exp(-ip\tau\hat{n}). \tag{11}$$

Equation (2) is now satisfied if and only if $\Sigma_{s=1}^{p}V(\theta-s\tau)=\beta/k$ and $\eta=\tau/2\pi=m/p$, for relatively prime integers m and p. The latter condition (rational η) is precisely the general condition for quantum resonance [4,23]. The former condition gives, however, the opposite phenomenon, i.e., the QAR. It is easy to see that this condition is satisfied only if the Fourier coefficients v_n of $V(\theta)$ satisfy

$$v_0=\frac{\beta}{pk}, \quad v_{sp}=0 \quad (s\neq 0).$$

In fact, v_{sp} $(s\neq 0)$ are precisely the Fourier coefficients that contribute to the width of a QE band in the case of m/p quantum resonance [23].

These examples show that in ordinary KR systems QAR may occur only if $V(\theta)$ satisfies some restrictive conditions. In the next section we shall introduce systems in which QAR occurs for *arbitrary* kicking potentials, at some $\eta=\eta_0$.

III. QAR IN MODULATED KR SYSTEMS AND ASYMPTOTIC EXPONENTIAL LOCALIZATION

We define the general modulated kicked rotor by the Hamiltonian

$$H=\frac{L^2}{2I}+\hat{k}V(\theta)\sum_{j=0}^{M-1} c_j\Delta_T(t-t_j), \tag{12}$$

where $V(\theta)$ is an analytic function of θ, c_j are arbitrary coefficients for $j=0,\ldots,M-1$, and

$$0\leqslant t_j<t_{j+1}\leqslant T, \quad t_0=0, \quad t_M=T.$$

The Hamiltonian (12) has the general form (1) with $f(t)=C(t)\Sigma_{j=0}^{M-1}\Delta_T(t-t_j)$, where $C(t)$ is a periodic function with period T, satisfying the M conditions $C(t_j)=c_j$. Thus (12) may be viewed as a generalized KR with M kicks at arbitrary times t_j within the basic period and modulated by the function $C(t)$. The classical map for (12) is given by

$$L_{s+1}=L_s-\hat{k}c_jV'(\theta_s),$$
$$\theta_{s+1}=\theta_s+[(t_{s+1}-t_s)/I]L_{s+1}, \tag{13}$$

where the integer s is uniquely decomposed as $s=rM+j$ (r is an integer and $j=0,\ldots,M-1$), $t_s=rMT+t_j$, $L_s=L(t=t_s-0)$, and $\theta_s=\theta(t=t_s-0)$. In general, the system (12) with (13) is classically nonintegrable and exhibits the transition from local to global chaos when $\hat{k}$ is increased, as in the ordinary KR case [25]; see an example in Fig. 1. The simple case of $M=2$, with $c_0=-c_1=1$ and $t_1=T/2$ (the ''two-sided'' KR), was studied in detail in Refs. [21,22]. This case may already be considered as an approximation of sinusoidal driving potentials corresponding to ac electromagnetic fields [26]. Better approximations should be achieved by using the Hamiltonian (12), with properly chosen coefficients c_j. The study in Refs. [21,22] will now be extended to the general case of (12).

The evolution operator for (12), from $t=-0$ to $t=T-0$, is

$$U=\prod_{j=0}^{M-1}\exp(-i\tau_j\hat{n}^2)\exp[-ic_jkV(\theta)], \tag{14}$$

where, for $j=0,\ldots,M-1$,

$$\tau_j=\frac{\hbar(t_{j+1}-t_j)}{2I}, \tag{15}$$

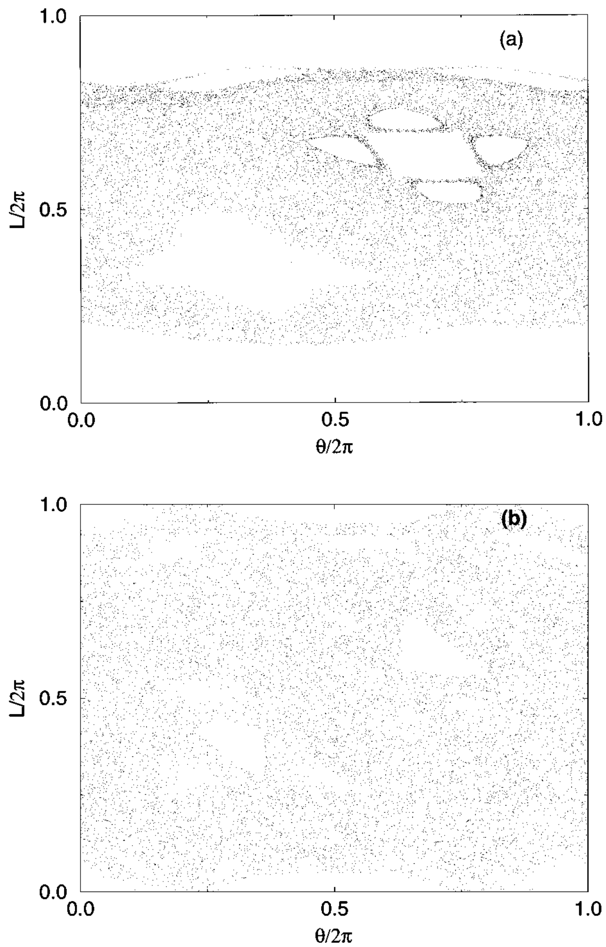

FIG. 1. Chaotic orbits generated by 30 000 iterations of the classical Poincaré map (13) for a $M=3$ Hamiltonian (12), with $V(\theta)=\cos(\theta)$, $c_0=c_1=1$, $c_2=-2$, and $t_{j+1}-t_j=T/3$ for all j: (a) local chaos for $\hat{k}=0.3$ and (b) global chaos for $\hat{k}=0.5$.

and the factors under the product sign in (14) are arranged from right to left in order of increasing j. Now, when all the quantities (15) are integer multiples of 2π, i.e., $\tau_j=2\pi m_j$, one has $\exp(-i\tau_j\hat{n}^2)=1$ identically. Then, if the coefficients c_j satisfy the condition

$$\sum_{j=0}^{M-1} c_j=0, \tag{16}$$

we find that $U=1$ in (14), corresponding to a simple case of fundamental QAR ($p=1$). For simplicity, we shall restrict ourselves in what follows to this case, characterized by the conditions $\tau_j=2\pi m_j$ and (16). More general cases, for both periods $p=1$ and $p=2$, are considered in Appendix A, where it is conjectured that QAR with period $p>2$ does not exist for the MKR Hamiltonian (12).

When $U=1$, the QE spectrum consists just of a single, infinitely degenerate level. The natural question is then precisely how this infinite degeneracy is removed by slightly perturbing η_j near their integer values m_j. For definiteness, the perturbation of τ_j in (15) will be made by perturbing I near $I=I_0$, leaving t_j fixed. Denoting by ϵ the corresponding perturbation in $\tau=\hbar T/2I$, the perturbation in τ_j is given by

$$\epsilon_j=\frac{t_{j+1}-t_j}{T}\epsilon. \tag{17}$$

Using the operator identity [27]

$$e^A B e^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]$$
$$+\frac{1}{3!}[A,[A,[A,B]]]+\cdots$$

and formally expanding the operators $\exp(-i\tau_j\hat{n}^2)=\exp(-i\epsilon_j\hat{n}^2)$ in powers of ϵ_j, we find, to first order in ϵ_j,

$$U\approx 1-\sum_{j=0}^{M-1}\epsilon_j\{i\hat{n}^2-d_jk[2iV'(\theta)\hat{n}+V''(\theta)]$$
$$+id_j^2k^2V'^2(\theta)\}, \tag{18}$$

where $d_j=\Sigma_{s=0}^{j}c_s$ and the prime on V denotes differentiation with respect to θ. Using (17), the expression in (18) can be written, to first order in ϵ, as $\exp(-i\epsilon G_1)$, where

$$G_1=[\hat{n}-k\overline{d}V'(\theta)]^2+k_{\text{eff}}^2V'^2(\theta) \tag{19}$$

and $k_{\text{eff}}=k\Delta d$. Here $\overline{d}$ and Δd are, respectively, the average and standard deviation of d_j with "probability distribution" $(t_{j+1}-t_j)/T$,

$$\overline{d}=\sum_{j=0}^{M-1}\frac{t_{j+1}-t_j}{T}d_j,\quad (\Delta d)^2=\sum_{j=0}^{M-1}\frac{t_{j+1}-t_j}{T}d_j^2-\overline{d}^2. \tag{20}$$

Using (20) and the definition of d_j, it is easy to show that k_{eff} (or Δd) is invariant under cyclic permutations of the sequence c_j [28].

Assuming for the moment the validity of the expansion above in powers of ϵ (see the discussion below), the QE states ψ in the limit of infinitesimal ϵ are precisely the eigenstates of G_1,

$$G_1\psi=g\psi, \tag{21}$$

with quasienergies $\omega=\epsilon g$ ($\epsilon\rightarrow 0$). Performing on (21) the gauge transformation

$$\varphi=\exp[-ik\overline{d}V(\theta)]\psi, \tag{22}$$

we obtain for φ, using (19), the eigenvalue equation

$$-\frac{d^2\varphi}{d\theta^2}+k_{\text{eff}}^2V'^2(\theta)\varphi=g\varphi. \tag{23}$$

We thus see that the QE problem for infinitesimal ϵ is just that of a Schrödinger equation (23) with a periodic potential. The spectrum g then has a band structure, but because of the periodic boundary condition $\varphi(2\pi)=\varphi(0)$, only the level with zero quasimomentum is picked out from each band. This gives, in general, a point spectrum. Now, being the solution of the linear differential equation (23), $\varphi(\theta)$ is analytic at least in the domain of analyticity of $V'(\theta)$ [29]. Let γ be the smallest distance of a singularity of $V'(\theta)$ from the real θ axis. Then the Fourier-series expansion of $\varphi(\theta)$ will converge at least within an infinite horizontal strip of width 2γ, symmetrically positioned around the real θ axis [29]. It follows that the Fourier coefficients of φ and ψ in (22) decay asymptotically at least as $\exp(-\gamma|n|)$. This means that in the immediate vicinity of QAR, *asymptotic exponential localization* takes place in the angular momentum $n\hbar$, with localization length ξ_0 not larger than $1/\gamma$. In general, ξ_0 is determined entirely by the analytical properties of $V'(\theta)$ (see examples in Sec. IV).

This exponential ''QAR localization'' in L, following from Eq. (23), justifies *a posteriori* the expansion above in powers of ϵ. In fact, the general expansion for U in (14) can be formally written as $\exp(-iG)$, $G=\Sigma_{j=1}^{\infty}\epsilon^j G_j$. Here the Hermitian operators G_j are polynomials in $\hat{n}$ and derivatives of $V(\theta)$ of order not larger than $2j$ [the leading operator G_1 is given by (19)]. Thus the highest power of $\hat{n}$ contributed by G_j appears in this expansion as $(\epsilon\hat{n}^2)^j$. This means that by choosing $\epsilon < n_{\rm max}^{-2}$, where $n_{\rm max} \gg 1/\gamma$, the eigenstates of U (i.e., the QE states) should be very close to those of G_1, at least within the localization domain. In the limit of infinitesimal ϵ, the QE states should coincide with the eigenstates of G_1. The derivation of Eq. (23) appears then to be self-consistent.

We now show that the effective Hamiltonian (19) has a classical counterpart in the limit of very small values of the quantity $T/I=2\tau/\hbar$. This limit corresponds to the case of infinitesimal values of τ_j in (15), i.e., infinitesimally close to the special QAR point $\eta=\tau/2\pi=0$ ($m_j=0$ for all j). Consider the Mth iteration of the classical map (13), giving the map $(L_s,\theta_s)\rightarrow(L_{s+M},\theta_{s+M})$. Taking carefully the limit $T\rightarrow 0$ in this map at fixed $(t_{j+1}-t_j)/T$ and using the condition (16), we obtain, after a straightforward but tedious calculation, the Hamilton equations

$$\frac{dL}{dt}=-\frac{\partial H_{\rm eff}}{\partial\theta},\quad \frac{d\theta}{dt}=\frac{\partial H_{\rm eff}}{\partial L}, \tag{24}$$

where

$$H_{\rm eff}=\frac{\hbar^2}{2I}G_1=\frac{1}{2I}[L-\hat{k}\overline{dV'}(\theta)]^2+\frac{(\hat{k}\Delta d)^2}{2I}V'^2(\theta). \tag{25}$$

Equations (24) and (25) show that the general MKR Hamiltonian (12), with coefficients c_j satisfying (16), is integrable in the limit $T\rightarrow 0$, as it reduces precisely to the 1D effective Hamiltonian (25). The latter is essentially the QAR effective Hamiltonian (19) and, after the canonical transformation $L'=L-\hat{k}\overline{dV'}(\theta)$ [analogous to the gauge transformation (22)], it becomes essentially the Schrödinger Hamiltonian in (23). Thus QAR localization in the infinitesimal vicinity of $\eta=0$ is associated with a classically integrable limit. As T is increased from 0, keeping the quantities $(t_{j+1}-t_j)/T$ fixed at some rational values m_j/m ($m=\Sigma_{j=0}^{M-1}m_j$), the QAR localization for $\eta=0$ will repeat periodically in the infinitesimal vicinity of $\eta=rm$, for all integers r. For these values of η, which are equivalent to $\eta=0$ but correspond to nonintegrable systems in a strong quantum regime, the QAR localization is only a ''reflection'' of the classically integrable limit $T\rightarrow 0$. In Appendix A, we show that QARs of periods $p=1$ and $p=2$ can occur, in general, if τ_j in (15) is an odd multiple of π. Such values of τ_j are *not* equivalent to $\tau_j=0$ since $\tau_j {\rm mod} 2\pi\neq 0$. In this case, the QAR localization is not even a reflection of a classically integrable limit.

It is important to notice that the limit $T\rightarrow 0$ (or $\eta\rightarrow 0$) at fixed $\hat{k}$ (or k) is *not* a semiclassical limit. In fact, if the quantities $(t_{j+1}-t_j)/T$ are kept fixed and the coordinate transformation $L'=(T/I)L$ is performed in the map (13), it becomes clear that the classical dynamics depends only on the parameter $K=(T/I)\hat{k}=2\tau k$. The semiclassical limit is then $\eta\rightarrow 0$ at fixed K, not at fixed $\hat{k}$. However, at fixed $k\gg 1$, small values of η such that $K\ll 1$ may be viewed as corresponding to a semiclassical regime of almost integrability.

IV. EXACTLY SOLVABLE CASES

In this section we consider cases of potentials $V(\theta)$ for which the QE problem in the infinitesimal vicinity of QAR [Eq. (23)] can be solved in closed form, or at least an explicit expression can be obtained for the QAR-localization length ξ_0. Using these exact results, we shall provide strong numerical evidence that the QE spectrum and localization features sufficiently close to QAR are well accounted for by the QAR effective Hamiltonian (19). As shown in Sec. V, the MKR dynamical problem can be mapped into pseudorandom tight-binding models, so that DL is expected to occur if $\eta=\tau/2\pi$ is sufficiently irrational. If, in addition, $\eta-\eta_0$ is small enough, this DL should look similar to QAR localization.

Our first example is the standard potential $V(\theta)=\cos(\theta)$, for which Eq. (23) reduces to the *Mathieu equation* [29,30]

$$y''+[a-2q\cos(2\theta)]y=0, \tag{26}$$

where $y=\varphi$, $a=g-(k\Delta d)^2/2$, and $q=-(k\Delta d)^2/4$. The problem is then exactly solved in terms of the periodic Mathieu functions $y=ce_r(\theta,q)$ (symmetric) and $y=se_r(\theta,q)$ (antisymmetric), with corresponding eigenvalues $a=a_r(q)$ and $a=b_r(q)$. Explicit expressions for these functions and eigenvalues, as well as a detailed discussion of their properties, can be found in Refs. [29,30]. From Eq. (22) the Fourier coefficients ψ_n and y_n of ψ and $y=\varphi$, respectively, are related by $\psi_n=\Sigma_j i^j J_j(kd)y_{n-j}$, where $J_j(kd)$ is a Bessel function. Since the dominant decay rate of both J_n and y_n with n is like n^{-n} [30], this is also the dominant decay rate of ψ_n. This strong localization in L space, faster than exponential, could be expected from the fact that $V(\theta)=\cos(\theta)$ is an entire function (analyticity-strip width $2\gamma=\infty$), so that the asymptotic localization length $\xi_0=1/\gamma=0$.

Let us now check to what extent the QAR effective Hamiltonian (19) for $V(\theta)=\cos(\theta)$ reproduces accurately the quantum dynamics and QE spectrum for $\eta-\eta_0=\epsilon/2\pi$ sufficiently small and irrational. We have studied numerically the case of $M=3$, with $c_0=c_1=1$, $c_2=-2$, and $\tau_j=\tau/3$ for all j (chaotic orbits for this system are shown in Fig. 1). The quantum dynamics of a wave packet initially equal to $|n=0\rangle$ was investigated using a basis of up to 512 angular-momentum states around $n=0$. The wave packet was propagated in time using well-known algorithms [6]. In Fig. 2, we

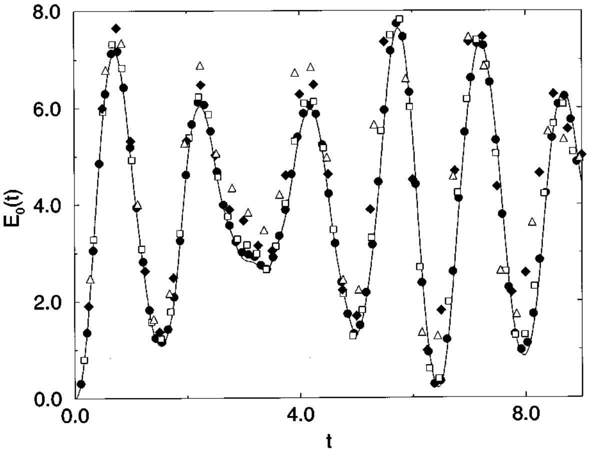

FIG. 2. Expectation value of the kinetic energy E_0 as a function of the ''real time'' $t=\epsilon s$, for the MKR system described in the caption of Fig. 1 with $k=2$ and for several values of $\epsilon=2\pi(\eta-\eta_0)$ (η_0 is a QAR point). The continuous curve corresponds to $\epsilon=0.01$, the filled circles to $\epsilon=0.11$, the squares to $\epsilon=0.17$, the filled diamonds to $\epsilon=0.25$, and the triangles to $\epsilon=0.28$.

plot the kinetic-energy expectation value $E_0=\langle L^2/2I\rangle$ as a function of the ''real time'' $t=\epsilon s$ (s is the number of applications of U), for $k=2.0$ ($k_{\rm eff}\equiv k\Delta d=\sqrt{8/3}$) and several irrational values of $\epsilon/2\pi$. We observe that almost all the data fall close to the same curve, even for values of ϵ as large as $\epsilon=0.28$. For smaller values of $k_{\rm eff}$, the data fall much more accurately on the same curve (see an example for $M=2$ in Ref. [22]). This is evidence that, even for ϵ not very small, the quantum dynamics over a significant time interval can be well described by the approximate evolution operator $\overline{U}^s=\exp(-i\epsilon sG_1)=\exp(-itG_1)$, generated by the QAR effective Hamiltonian G_1. In Fig. 3, we plot the Fourier transform $E_0(\nu)$ of $E_0(t)$ for $\epsilon=0.1$ and several values of k. The positions of the various peaks in $E_0(\nu)$ must correspond to

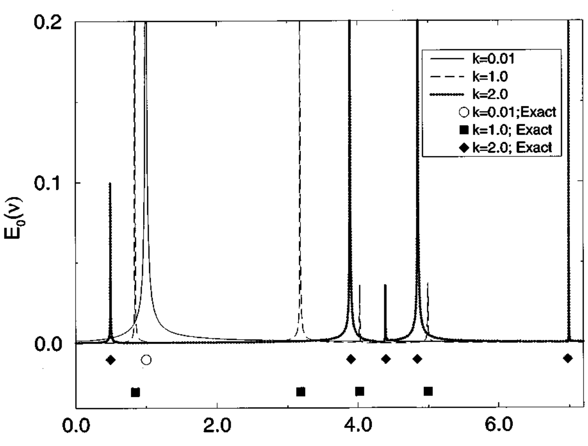

FIG. 3. Fourier transform $E_0(\nu)$ of $E_0(t)$ (see the caption of Fig. 2) for $\epsilon=0.1$ and several values of k (see the legend). The symbols at the bottom are the theoretical predictions for the peaks positions, based on the eigenvalues of the Mathieu equation (26). The peaks for $k=0.01$ and $k=1.0$ have been rescaled by a factor of 100 000 and 10, respectively, for visibility.

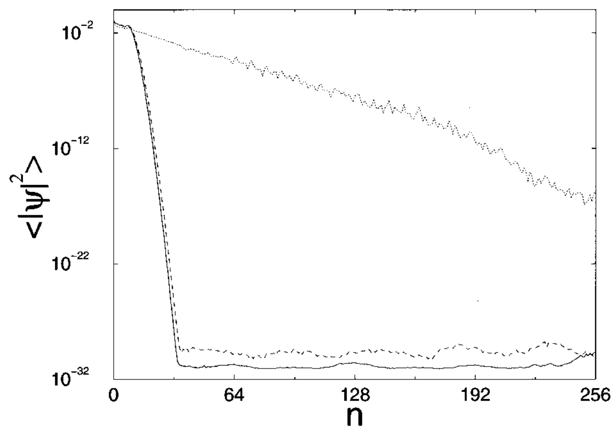

FIG. 4. Steady-state probability distributions $\langle|\psi(n)|^2\rangle$ over the angular momentum $n\hbar$ for the $M=2$ MKR ($c_0=-c_1=1$ and $t_1=T/2$) with $V(\theta)=\cos(\theta)$ and $k=10$. The three curves correspond to three irrational values of $\eta=\tau/2\pi$: $\tau=10^{-5}$ (solid line), $\tau=10^{-3}$ (dashed line), and $\tau=10^{-1}$ (dotted line). The saturation of $\langle|\psi(n)|^2\rangle$ around 10^{-30} is due to numerical noise.

values of $\epsilon\nu$ equal to the spacings between QE levels. A comparison with the level spacings corresponding to eigenvalues of the Mathieu equation (26) shows excellent agreement. This is strong evidence that sufficiently close to QAR the QE spectrum is very well accounted for by the QAR operator G_1 [i.e., Eq. (23)].

Dynamical localization [17] in the vicinity of QAR will now be compared with QAR localization (we assume in what follows that $\eta_0=0$). For this purpose, it will be sufficient to consider the localization of steady-state probability distributions $\langle|\psi(n)|^2\rangle$ over angular momentum $n\hbar$. Such a distribution is calculated by propagating an initial angular-momentum state $|n=0\rangle$ for a sufficiently long time interval and averaging then the results for $|\psi(n)|^2$ over a subinterval at the end of this interval. All our numerical calculations in what follows have been restricted to the simple case of the $M=2$ MKR with $c_0=-c_1=1$ and $t_1=T/2$ (the ''two-sided'' KR [21,22]). Figure 4 shows a semilogarithmic plot of $\langle|\psi(n)|^2\rangle$ for $V(\theta)=\cos(\theta)$, $k=10$, and three irrational values of $\eta=\tau/2\pi$, $\tau=10^{-5},10^{-3},10^{-1}$. For the first two values of τ, corresponding to close vicinity of QAR, $\langle|\psi(n)|^2\rangle$ appears to decay faster than exponential, in accord with the solutions of the Mathieu equation (26) (i.e., QAR localization). On the other hand, the localization for $\tau=10^{-1}$ looks quite different from QAR localization; it appears, in fact, to be almost exponential. These results can be explained as follows. The classical nonintegrability parameter is given by $K=(T/I)\hat{k}=2\tau k$ (see Sec. III). Thus the first two values of τ correspond to a semiclassical regime (small τ) of almost integrability (small K) and the observed DL [17] is then similar to QAR localization, as predicted from the integrable effective Hamiltonian (19) (however, a basic difference between these two localizations is expected, as pointed out in Sec. VI). For $\tau=10^{-1}$, on the other hand, $K=2$, corresponding to a semiclassical regime (small τ) of global chaotic diffusion in the $M=2$ MKR [21]. As in the KR case [7,8], the localization observed here should be characterized by a DL length ξ determined by the chaotic-diffusion coefficient D, $\xi\approx D/2$ (see Sec. VI). This is in contrast with QAR localization, which is completely determined by the analytical properties of the potential. The transition from QAR localization to DL, as τ is varied in a semiclassical regime at fixed k, will be studied in more detail in Sec. VI.

Our second example is the potential

$$V(\theta)=A\arctan[\kappa\cos(\theta)-\kappa_0], \tag{27}$$

where A, κ, and κ_0 are some constants. The QAR-localization length ξ_0 for (27) can be determined exactly as follows. The function $V'(\theta)$ assumes simple poles θ_0 satisfying the equation $\kappa\cos(\theta_0)-\kappa_0=\pm i$. The distance of any of these poles from the real θ axis is $\gamma=|\mathrm{Im}(\theta_0)|$, and we easily find that

$$2\kappa\cosh(\gamma)=[1+(\kappa_0+\kappa)^2]^{1/2}+[1+(\kappa_0-\kappa)^2]^{1/2}. \tag{28}$$

Consider now the Fourier-series expansion for the solutions $\varphi(\theta)$ of Eq. (23). We claim that in the case of (27) the Fourier coefficients φ_n must decay *asymptotically* as $\varphi_n\asymp\exp(-\gamma|n|)$. The QAR-localization length is then $\xi_0=1/\gamma$. To show this, we observe that the simple poles θ_0 correspond to *regular singularities* [29] of order 2 of Eq. (23). The exponents ρ_1 and ρ_2 for these singularities are easily determined from the quadratic ''indicial'' equation [29] for (23): $\rho_1=\rho_2=1/2$. Since the exponents are equal, the general solution of (23) around $\theta=\theta_0$ assumes the form [29]

$$\varphi(\theta)=(\theta-\theta_0)^{1/2}\{R_1(\theta-\theta_0)+R_2(\theta-\theta_0)\times[b_1+b_2\ln(\theta-\theta_0)]\}, \tag{29}$$

where $R_1(\theta)$ and $R_2(\theta)$ are analytic (can be expressed as Taylor expansions) around $\theta=\theta_0$ and b_1 and b_2 are arbitrary constants. It follows from (29) that all the derivatives of $\varphi(\theta)$ diverge at $\theta=\theta_0$. Let us now continue the Fourier-series expansion for $\varphi(\theta)$ into the complex θ plane. Defining the complex variable $z=e^{i\theta}$, one gets a Laurent expansion in z that converges at least in a ''ring'' excluding the singularities of Eq. (23) [29], i.e., for $|\mathrm{Im}(\theta)|<\mathrm{Im}(\theta_0)=\gamma$. However, since all the derivatives of (29) diverge as $\theta\to\theta_0$, this must be also the case for the derivatives of the Laurent expansion. By a simple consideration of the latter derivatives, the desired relation $\varphi_n\asymp\exp(-\gamma|n|)$ is obtained.

Figures 5 and 6 show semilogarithmic plots of $\langle|\psi(n)|^2\rangle$ for the $M=2$ MKR (defined as above) with the potential (27) ($A=1$, $\kappa=1$, and $\kappa_0=0$) and for several values of k and τ. The straight thick line in both figures has slope -2γ, where γ is determined from Eq. (28). For τ sufficiently small, this slope is expected to be close to the asymptotic rate of exponential decay of $\langle|\psi(n)|^2\rangle$. We see that this is indeed the case whenever the classical parameter $K=2\tau k$ is small enough, $K<1$, corresponding to almost integrability or local chaos. In particular, for $k=2$ (Fig. 5) the decay rate appears to be equal to 2γ for all three values of τ. In fact, for $k=2$ and $A=1$ the potential (27) leads to a nearest-neighbor pseudorandom tight-binding model whose DL length ξ appears to be independent of τ (see Sec. VI and Fig. 8). On the other hand, for $k=10$ (Fig. 6) the DL length is quite sensitive to the value of τ. In the case of $\tau=10^{-1}$, corresponding to a semiclassical regime of global chaotic diffusion ($K=2$), the observed DL looks quite different from QAR localization, with an asymptotic DL length $\xi\approx D/2\neq\xi_0$. This case is similar to that of $\tau=10^{-1}$ in Fig. 4.

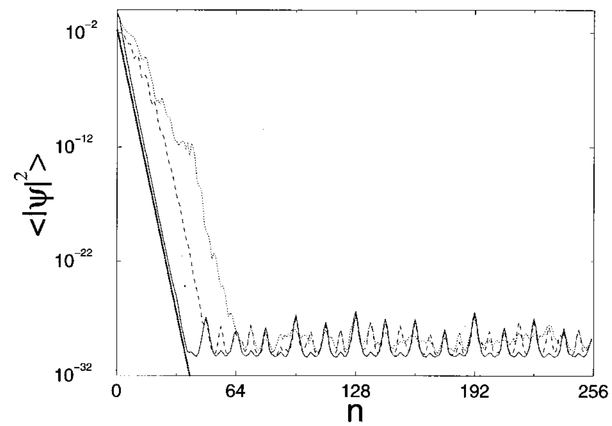

FIG. 5. Same as in Fig. 4, but for the potential (27) with $A=1$, $\kappa=1$, $\kappa_0=0$, and $k=2$. The straight thick line has slope -2γ, where γ is determined from Eq. (28). For τ sufficiently small, this slope is expected to be very close to the asymptotic rate of exponential decay of $\langle|\psi(n)|^2\rangle$. We observe that this is the case even for τ not very small, $\tau=10^{-1}$ (dotted line). In fact, this system corresponds precisely to a pseudorandom Lloyd model, whose DL length ξ seems to be independent of τ (see Sec. VI and Fig. 8).

V. MULTICHANNEL PSEUDORANDOM TIGHT-BINDING MODELS

We now show how the MKR dynamical problem with (12) can be mapped into a pseudorandom tight-binding model, in analogy to the ordinary KR case [6]. For simplicity, we shall assume that the quantities (15) are all equal, $\tau_j=\tau/M$ for all j (i.e., the kicks are equidistant in time). Let $u_j^{\pm}(\theta)$, $j=0,\ldots,M-1$, denote a QE state with quasienergy ω at time $t=t_j\pm0$. The following relations hold:

$$u_j^{+}(\theta)=\exp[-ic_jkV(\theta)]u_j^{-}(\theta), \tag{30}$$

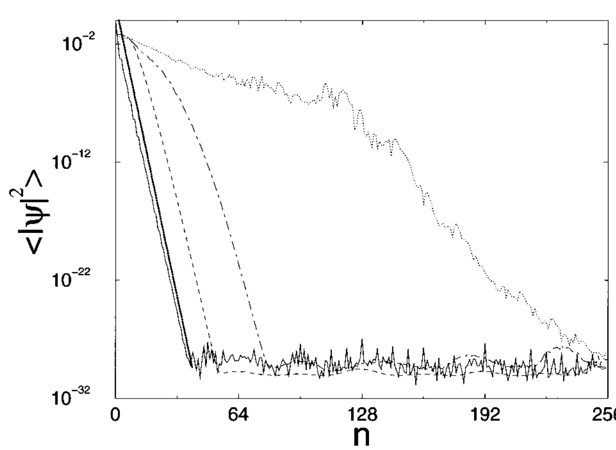

FIG. 6. Same as in Fig. 5, but for $k=10$. The four curves correspond to four irrational values of $\eta=\tau/2\pi$: $\tau=10^{-7}$ (solid line), $\tau=10^{-5}$ (dashed line), $\tau=10^{-3}$ (dot-dashed line), and $\tau=10^{-1}$ (dotted line). The straight thick line has slope -2γ, where γ is determined from Eq. (28).

$$u_{j,n}^{-}=e^{-i\tau n^2/M}u_{j-1,n}^{+} \quad (0<j\leqslant M-1),$$

$$u_{0,n}^{-}=e^{i(\omega-\tau n^2/M)}u_{M-1,n}^{+}, \tag{31}$$

where $u_{j,n}^{\pm}$ is the L representation of $u_j^{\pm}(\theta)$. We define, in some analogy with Ref. [6],

$$u_j(\theta)=e^{ij\omega/M}\frac{u_j^{+}(\theta)+u_j^{-}(\theta)}{2}, \quad e^{-ic_jkV(\theta)}=\frac{1+iW_j(\theta)}{1-iW_j(\theta)}, \tag{32}$$

so that $W_j(\theta)=-\tan[c_jkV(\theta)/2]$. Simple manipulations of relations (30)–(32) yield a system of M equations for the L representation $u_{j,n}$ of $u_j(\theta)$, $j=0,\ldots,M-1$:

$$u_{j+1,n}-i\sum_r W_{j+1,n-r}u_{j+1,r}$$

$$=e^{i(\omega-\tau n^2)/M}\left(u_{j,n}+i\sum_r W_{j,n-r}u_{j,r}\right), \tag{33}$$

where $u_{M,n}\equiv u_{0,n}$ and $W_{j,n}$ is the L representation of $W_j(\theta)$. Unless otherwise specified, the index r in (33) runs over all the integers. We now introduce the Fourier transforms $\overline{u}_{s,n}$ and $\overline{W}_{s,n}$ of $u_{j,n}$ and $W_{j,n}$, respectively, in the variable j:

$$\overline{u}_{s,n}=\frac{1}{M}\sum_{j=0}^{M-1} u_{j,n}e^{2\pi ijs/M}, \tag{34}$$

$$\overline{W}_{s,n}=\frac{1}{M}\sum_{j=0}^{M-1} W_{j,n}e^{2\pi ijs/M}. \tag{35}$$

Using the expressions (34) and (35) in (33), we obtain, after simple algebraic manipulations,

$$T_n^{(s)}\overline{u}_{s,n}+\sum_{r\neq n}\overline{W}_{0,n-r}\overline{u}_{s,r}+\sum_r\sum_{s'\neq s}\overline{W}_{s-s',n-r}\overline{u}_{s',r}=E\overline{u}_{s,n}, \tag{36}$$

where, for $s=0,\ldots,M-1$,

$$T_n^{(s)}=-\tan[(\tau n^2-2\pi s-\omega)/2M], \tag{37}$$

the index s' takes all the integer values $s'=0,\ldots,M-1$ with the exception of $s'=s$, and $E=-\overline{W}_{0,0}$. Equations (36) describe a tight-binding model of an M-channel strip [20]. The on-site potential in channel s is given by $T_n^{(s)}$, while the hopping constants within a channel are $\overline{W}_{0,n}$. The hopping constants from channel s to channel $s'\neq s$ are given by $\overline{W}_{s-s',n}$.

A particularly interesting case arises when c_j assumes only the values $0,\pm c$ for some constant c, which will be chosen, without loss of generality, equal to 1. From relation (35) and the definition $W_j(\theta)=-\tan[c_jkV(\theta)/2]$, it follows then that

$$\overline{W}_{s,n}=c(s)W_n\equiv\frac{1}{M}\sum_{j=0}^{M-1} c_je^{2\pi ijs/M}W_n, \tag{38}$$

where $c(s)$ is defined by (38) and W_n is the L representation of $W(\theta)\equiv-\tan[kV(\theta)/2]$. Relations (38) and (16) imply now that the hopping constants $\overline{W}_{0,n}$ within a channel are identically zero, including, of course, $E=-\overline{W}_{0,0}$. In this *singular* case, the model (36) loses much of its physical meaning. In Appendix C, we consider in some detail nearly singular cases of (36), for which the hopping constants $\overline{W}_{0,n}$ are nonzero but small. Notice that the $M=2$ case is always singular, since $c_0=-c_1$ from (16). In this case, however, a physically meaningful two-channel model [22] can be derived directly from Eq. (33), without the need of the Fourier transforms (34) and (35). Simple manipulations of Eq. (33) give in this case

$$T_nu_{0,n}+S_nu_{1,n}+\sum_{r\neq 0} W_{n-r}u_{0,r}=Eu_{0,n},$$

$$-T_nu_{1,n}-S_nu_{0,n}+\sum_{r\neq 0} W_{n-r}u_{1,r}=Eu_{1,n}, \tag{39}$$

where $T_n=\cot(\zeta_n)$, $S_n=-1/\sin(\zeta_n)$, $\zeta_n=(\tau n^2-\omega)/2$, and $E=-W_0$. The on-site potential and hopping constants within each channel are, respectively, T_n and W_n, while S_n are the coupling constants between the channels. As shown in Sec. VI, the $M=2$ MKR is essentially equivalent to an ordinary KR if the potential satisfies $V(\theta+\pi)=-V(\theta)$. In this case, the dynamical problem can be conveniently approached using the well-known tight-binding models for the KR [6,7].

For irrational $\eta=\tau/2\pi$, $T_n^{(s)}$ in (37) [or T_n and S_n in (39)] is a pseudorandom sequence [6,15] that, by arguments similar to those used in the KR case [6], may lead to Anderson-like localization of the eigenstates of (36) [or of (39)]. This is the DL [17] of the corresponding QE states in angular momentum. Another source of randomness, which does not involve η but may contribute significantly to DL, is the dependence of the hopping constants $\overline{W}_{s,n}$ on the distance s between channels, especially in the limit of very large M. This may reflect a possible randomness of the sequence c_j in the ''time index'' j.

Clearly, the pseudorandomness is not defined in the infinitesimal vicinity of QAR (infinitesimal $\epsilon=\tau\,\mathrm{mod}\,2\pi$), so that QAR localization is not DL, strictly speaking. However, when approaching a QAR point η_0, the pseudorandomness is guaranteed by choosing $\epsilon/2\pi$ in a sequence of ''strong'' irrationals $\epsilon_l/2\pi$, $l=1,2,\ldots$, converging to 0 [e.g., $\epsilon_l=2\pi/(l+\varrho)$, where ϱ is the golden mean]. It is then quite possible that for $\epsilon=\epsilon_l$ the model (36) [or (39)] will have exponentially localized eigenstates. In fact, the numerical evidence presented in Sec. IV (see also Sec. VI) strongly indicates that this is indeed the case for l sufficiently large. Then, provided the QAR condition (16) is satisfied, the quasienergy ω in (37) should be given approximately by $\omega=\epsilon g$, where g is an eigenvalue of the leading operator G_1 in (19). Moreover, in the $M=2$ case of (39), one can easily determine, using (30), (32), and (22), the accurate relation expected between the solutions of Eqs. (23) and (39) for small ϵ: $u_0(\theta)\approx\cos[kV(\theta)/2]\varphi(\theta)$.

For rational values of $\eta=m/p$ (m and p are relatively prime integers), the on-site potential $T_n^{(s)}$ in (37) [or T_n in

(39)] is periodic in n with period p. Then, by straightforward application of Bloch theorem, one can easily show that the QE spectrum ω is, in general, absolutely continuous, consisting of p bands. This corresponds to usual quantum resonance [4]. However, when the QAR conditions are satisfied, either for $p=1$ or $p=2$ [see (16) and Appendix A], a QE band ''collapses'' to an infinitely degenerate level.

VI. TRANSITION FROM QAR LOCALIZATION TO DYNAMICAL LOCALIZATION IN A STRONG-CHAOS REGIME

In this section, we study the dependence of the DL length ξ on the distance $\eta-\eta_0$ from the QAR point $\eta_0=0$. This study will be performed in a natural way at fixed k, so that $\eta_0=0$ is associated with a classically integrable limit (see Sec. III). If k is sufficiently large, the classical parameter $K=2\tau k=4\pi\eta k$ may correspond to a case of global chaotic diffusion already for small values of $\eta<1$, i.e., in the semiclassical regime. The approximate relation $\xi\approx D/2$ [7,8], where D is the diffusion coefficient [see definition (49) below], should then hold. The transition from QAR localization to DL in a strong-chaos regime, as η is increased from 0, can be understood in a most illuminating way if the problem is approached by using a suitable pseudorandom tight-binding model, equivalent to the dynamical system. This approach is also most convenient for an accurate calculation of ξ.

For simplicity, we shall restrict ourselves to the $M=2$ MKR ($c_0=-c_1=1$, $t_1=T/2$) with potentials satisfying $V(\theta+\pi)=-V(\theta)$ (as in the numerical examples in Figs. 4–6). We denote by $U_{2\mathrm{KR}}(\tau)$ the evolution operator for this system at a given value of τ. Similarly, we denote by $U_{\mathrm{KR}}(\tau)$ the evolution operator (6) for the ordinary KR. Using relations (14) and (7) with $V(\theta+\pi)=-V(\theta)$, it is easy to derive the exact relation

$$U_{\mathrm{KR}}^2(\pi+\epsilon/2)=U_{2\mathrm{KR}}(\epsilon). \tag{40}$$

Relation (40) means that the quantum dynamics of the $M=2$ MKR at distance ϵ from the QAR point $\tau=0$ is essentially equivalent to that of the ordinary KR at distance $\epsilon/2$ from $\tau=\pi$. The latter value of τ is precisely a *period-2* QAR point for the KR (see Sec. II and Appendix A). This equivalence between the two systems enables one to study the $M=2$ MKR using the simple tight-binding models for the KR [6,7], instead of the two-channel model (39).

Our basic potential is the standard one, $V(\theta)=\cos(\theta)$. A convenient tight-binding model for the KR with this potential was proposed by Shepelyansky [7]:

$$\sum_{r=-\infty}^{\infty} J_r(k/2)\sin[(\tau n^2-\omega+\pi r)/2]\overline{u_{n+r}}=0, \tag{41}$$

where $J_r(k/2)$ is the Bessel function, ω is the quasienergy, and $\overline{u_n}$ is related to the angular-momentum representation of the QE states. Using relation (40) in (41), we obtain the corresponding tight-binding model for the $M=2$ MKR:

$$\sum_{r=-\infty}^{\infty} J_r(k/2)\sin\{[(2\pi+\tau)n^2-\omega+2\pi r]/4\}\overline{u_{n+r}}=0, \tag{42}$$

where all the quantities τ, ω, and $\overline{u_n}$ now refer to the $M=2$ MKR. Let us briefly recall how the asymptotic DL length ξ can be calculated from such tight-binding models [7,8]. The hopping constants in (42), given by the Bessel function $J_r(k/2)$, decay faster than exponentially for $|r|>k/2$:

$$J_r(k/2)\sim\begin{cases}(4/\pi k)^{1/2}\cos(k/2-\pi r/2-\pi/4) & \text{for } |r|<k/2\\ (1/2\pi|r|)^{1/2}(ek/4r)^{|r|} & \text{for } |r|>k/2.\end{cases} \tag{43}$$

It is therefore reasonable to approximate (42) by restricting r to the finite range $|r|\leqslant N$, for sufficiently large N. This truncated form of Eq. (42) can be easily written as a transfer-matrix problem [7,8]

$$\sigma_{s+1}=\Gamma_s\sigma_s, \tag{44}$$

where σ_s is the $2N$-dimensional vector with components $\sigma_s^{(r)}=\overline{u_{s-r}}$, $r=-N+1,\ldots,N$, and Γ_s is a $2N\times 2N$ symplectic matrix. One may interpret (44) as a map describing a Hamiltonian dynamical system with N degrees of freedom [18]. The vector σ_0 is mapped into σ_n, for arbitrary $n>0$, by the product matrix

$$\Lambda_n=\Gamma_{n-1}\Gamma_{n-2}\cdots\Gamma_0. \tag{45}$$

Since the matrix (45) is, obviously, symplectic, its eigenvalues $\lambda(n)$ come in N reciprocal pairs $[\lambda_r(n),\lambda_r^{-1}(n)]$, $r=1,\ldots,N$, and we can always assume the ordering $1\leqslant|\lambda_1(n)|\leqslant|\lambda_2(n)|\leqslant\cdots|\lambda_N(n)|$. The *minimal Lyapunov exponent* for the map (44),

$$\gamma_N=\lim_{n\to\infty}\frac{1}{n}\ln|\lambda_1(n)|, \tag{46}$$

determines then an Nth-order approximation $\xi_N=1/\gamma_N$ to the asymptotic DL length ξ. Since an accurate calculation of γ_N becomes extremely time consuming as N is increased, important questions are how fast ξ_N converges to its limit value ξ as N is increased and whether ξ_N has a well-defined quantum-dynamical meaning *per se*.

We have therefore studied γ_N as a function of both N and η near QAR. The value of k was fixed at $k=10$ and several values of τ were considered in the interval $10^{-6}\leqslant\tau\leqslant 2$. We have calculated γ_N for $N\leqslant 15$, using the well-known method [7,8,18] for determining the Lyapunov spectra of products of matrices such as (45). The method is based on direct application of the map (44) a large number $n=n_{\max}$ of times, such that for $n\sim n_{\max}$ the matrices Γ_n can be considered as random. This randomness should be realized to some extent by the pseudorandom term $\tau n^2/4$ in (42) if $\tau[(n+1)^2-n^2]/4\sim 2\pi$ or $n=n_{\max}\sim 4\pi/\tau$. In practice, it was sufficient to use $n_{\max}\leqslant 10^6$ for all the values of τ considered. We have checked that this choice of $n_{\max}$ yields well-converged results by calculating and comparing γ_N for different values of $n\leqslant n_{\max}$, e.g., $n=10^5$, $2\times 10^5,\ldots,10^6$ for $n_{\max}=10^6$. The final results are shown in Fig. 7. We observe that γ_N clearly decreases in the interval $N\leqslant 3$ for all values of τ. For larger values of N and for very small τ (immediate vicinity of QAR), γ_N appears to increase without bounds

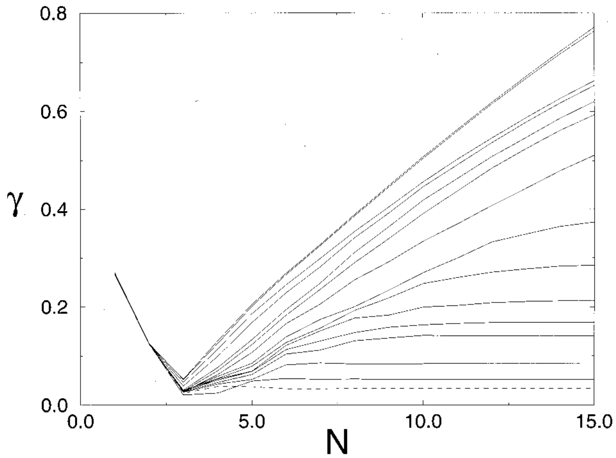

FIG. 7. Minimal Lyapunov exponent γ_N associated with the Nth-order truncation of the tight-binding model (42) (i.e., r restricted to $|r|\leqslant N$) with $k=10$. This model is equivalent to the $M=2$ MKR in Fig. 4. The several curves correspond, in descending order at $N=15$, to the following values of τ in (42): $\tau=10^{-6}$, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 2×10^{-2}, 5×10^{-2}, 0.1, 0.125, 0.15, 0.175, 0.2, 0.25 1.0, and 2.0 (dashed line). Notice the near coincidence of the curves for $\tau=10^{-6}$ and $\tau=10^{-5}$, indicating very close proximity to QAR. See details concerning the calculation of γ_N in the text.

with N. For $\tau\geqslant0.1$, γ_N appears to ''saturate'' around some limit value that decreases as τ is increased. For $\tau\sim1$, the saturation occurs almost immediately after $N>3$.

The apparently unbounded increase of γ_N with N for very small τ is consistent with the fact that the asymptotic QAR-localization length for $V(\theta)=\cos(\theta)$ is $\xi_0=0$ (analyticity-strip width $2\gamma=2\gamma_\infty=\infty$). We now show that γ_N has a well-defined quantum-dynamical meaning also for finite N. This is because the truncated version of the tight-binding model (42) turns out to be exactly equivalent to the dynamical problem (KR or $M=2$ MKR) with a potential $V_N(\theta)$ replacing $V(\theta)=\cos(\theta)$. Then $\xi_N=1/\gamma_N$ is the DL length for this potential, reducing to the QAR-localization length $\xi_{N,0}$ in the limit $\tau\to0$. For very small τ (e.g., $\tau=10^{-6}$ and $\tau=10^{-5}$ in Fig. 7), ξ_N should be an excellent approximation to $\xi_{N,0}$. The potential $V_N(\theta)$ can be easily determined from the g-function approach of Shepelyansky [7]:

$$W_N(\theta)=g_N(\theta)\exp[-ikV_N(\theta)/2]=\sum_{r=-N}^{N}J_r(k/2)e^{ir(\theta-\pi/2)}, \tag{47}$$

where $g_N(\theta)$ is some real function. Solving Eq. (47) for $V_N(\theta)$, we obtain

$$V_N(\theta)=-\frac{2}{k}\arctan\left[\frac{\sum_{r=-N}^{N}J_r(k/2)\sin(r\theta-r\pi/2)}{\sum_{r=-N}^{N}J_r(k/2)\cos(r\theta-r\pi/2)}\right]. \tag{48}$$

Using $J_{-r}(k/2)=(-1)^rJ_r(k/2)$, it is easy to see that (48) satisfies $V_N(\theta+\pi)=-V_N(\theta)$, so that relation (40) between the KR and the $M=2$ MKR holds for the potential $V_N(\theta)$. For $N=1$, (48) reduces to (27) with $\kappa_0=0$ (see below). The QAR-localization length for (48) can be determined as in the case of the potential (27). It is easily shown that $V_N'(\theta)$ assumes simple poles θ_0 satisfying the equation

$$W_N(\theta_0)=\sum_{r=-N}^{N}J_r(k/2)e^{ir(\theta_0-\pi/2)}=0,$$

or $g_N(\theta_0)=0$, since the exponential function in (47) can never vanish. The QAR-localization length for $V_N(\theta)$ is given by $\xi_{N,0}=1/\gamma_N^{(0)}$, where $\gamma_N^{(0)}$ is the smallest distance of a pole θ_0 from the real θ axis. It is also the half-width of the strip of analyticity of $V_N(\theta)$ (Fourier-series representation). Since $\gamma=\infty$ for $V(\theta)=\cos(\theta)$, $\lim_{N\to\infty}\gamma_N^{(0)}=\infty$, so that $\gamma_N^{(0)}$ must generally increase with N, as is clear from Fig. 7 for very small τ.

We thus see that in the limit $\tau\to0$, γ_N is both the minimal Lyapunov exponent of a $2N\times2N$ symplectic matrix and the half-width of analyticity of the potential $V_N(\theta)$ for the truncated problem. As N increases, the region of analyticity of $V_N(\theta)$ increases without bounds and the QAR-localization length $\xi_{N,0}\to0$. With this in mind, we now consider the cases where γ_N appears to saturate for N not too large in Fig. 7. As in the KR case [7,8], we expect that in the global-chaos regime the saturated value of the DL length $\xi_N=1/\gamma_N$ is given approximately by $\xi\approx D/2$, where D is the classical chaotic diffusion coefficient. In complete analogy with Refs. [3,7], we shall define D for the MKR map (13) as

$$D=\lim_{r\to\infty}\frac{\langle(L_{rM}-L_0)^2\rangle}{rM}, \tag{49}$$

where $\langle\,\rangle$ denotes an average over an ensemble of initial conditions $\{(L_0,\theta_0)\}$. In a strong-chaos regime, we may assume, as usual, that the angles θ_s are independent and random, giving vanishing force-force correlations $\langle V'(\theta_s)V'(\theta_{s'})\rangle=0$. Using then (13) in (49), we obtain the following expression for D:

$$D=\frac{k^2}{M}\lim_{r\to\infty}\frac{1}{r}\left\langle\left[\sum_{r'=0}^{r-1}\sum_{j=0}^{M-1}c_jV'(\theta_{r'M+j})\right]^2\right\rangle$$
$$=k^2\langle c^2\rangle\int_0^{2\pi}V'^2(\theta)\frac{d\theta}{2\pi}, \tag{50}$$

where $\langle c^2\rangle=\sum_{j=0}^{M-1}c_j^2/M$ and, as in Ref. [7], we use units such that $\hbar=1$, giving $\hat{k}=k$. For the $M=2$ MKR, $\langle c^2\rangle=1$ and, with $k=10$ and $V(\theta)=\cos(\theta)$, we find from (50) that $D=50$. The minimal saturated value of γ_N in Fig. 7 is $\gamma_N\approx0.034$ (dashed line), so that the maximal DL length is $\xi\approx29\approx D/2$. We have checked that the approximated relation $\xi\approx D/2$ holds also for other values of k and τ.

We observe that the minimal saturated value of $\gamma_N\approx2/D$ is attained almost immediately after the decrease of γ_N in the interval $N\leqslant3$. This behavior was observed also for other values of k, with N in the interval $N<k/2$. The decrease in γ_N may be due to the fact that for $|r|<k/2$ the hopping constants (43) are of the same order of magnitude, so that the effect of each additional hopping constant is to

increase the DL length. For $|r|>k/2$, on the other hand, the hopping constants (43) decay faster than exponentially and their effect on γ_N turns out to be negligible in the strong-chaos regime, as shown in Fig. 7. This fact was used in Ref. [7] to justify the truncation of the tight-binding model for the KR at $N\approx k/2$.

We see in Fig. 7, however, that the effect of the exponentially small hopping constants for $|r|>k/2$ is *not negligible* in regimes where the chaos is not strong enough. In sharp contrast with the case of $N<k/2$, these small hopping constants have the somehow paradoxical effect to *increase* γ_N, i.e., to *decrease* the localization length ξ. This effect appears to continue up to a maximal value N^* of N. For $N>N^*$, γ_N saturates to a value approximately equal to γ_{N*}. Both N^* and γ_{N*} increase as τ or k decrease.

While we are unable to provide at this point a quantitative explanation of these phenomena, it will become apparent from the following discussion that they are quite natural. We already know that the truncated tight-binding model is exactly equivalent to a dynamical problem with a potential $V_N(\theta)$ [see (48)]. The region of analyticity $\mathcal{R}_N$ of $V_N(\theta)$ increases with N for N sufficiently large $(N>k/2)$. The strip of analyticity $\mathcal{R}_{\mathrm{QE},N}$ of the QE states has a width $2\gamma_N$, where γ_N depends on τ and determines the rate of exponential decay of the QE states in angular-momentum space. In the infinitesimal vicinity of QAR, $\mathcal{R}_{\mathrm{QE},N}=\mathcal{R}_N$, so that an increase in the analyticity of the potential results in a corresponding increase of the analyticity of the QE states. For finite τ, however, the increase of $\mathcal{R}_N$ leads to an increase of $\mathcal{R}_{\mathrm{QE},N}$ only up to $N\sim N^*$, where γ_N saturates to a value γ_{N*}. As the strength of chaos is increased by increasing τ (or K), γ_{N*} decreases (the DL length ξ increases). At the same time, N^* decreases, so that the influence of the analyticity of $V_N(\theta)$ on the analyticity of the QE states is gradually reduced. The extreme case corresponds to the strong-chaos regime. Here the increase of $\mathcal{R}_N$ does not lead to any increase in $\mathcal{R}_{\mathrm{QE},N}$ and γ_{N*} is totally determined by the diffusion coefficient D. This case of DL in a semiclassical regime of global chaos is thus completely different in nature from that of QAR localization. We may now say that the transition between these two kinds of quantum localization takes place via *a gradual reduction of the influence of the analyticity of the system on that of the QE states, as the level of chaos is increased*.

This characterization of the transition is quite natural as it has a well-known classical analogue. When a nonintegrability parameter is increased, the analyticity of functions representing classical structures (e.g., Kol'mogorov-Arnol'd-Moser tori and periodic orbits) generally decreases. A famous example is the golden-mean torus in the standard map. When the parameter K approaches from below the critical value $K_c\approx 0.9716$ for the disappearance of this torus, the width Δ_{GM} of the strip of analyticity of functions representing the torus shrinks to zero as K_c-K (for K sufficiently close to K_c) [31]. At $K=K_c$, the torus is a continuous nondifferentiable curve and for $K>K_c$ it becomes a cantorus allowing for global chaotic diffusion. The decrease of Δ_{GM} as K is increased is analogous to the decrease of γ_{N*}. Unlike Δ_{GM}, however, the minimal value of γ_{N*}, in the strong-chaos regime, is not zero but only inversely proportional to $D\gg 1$.

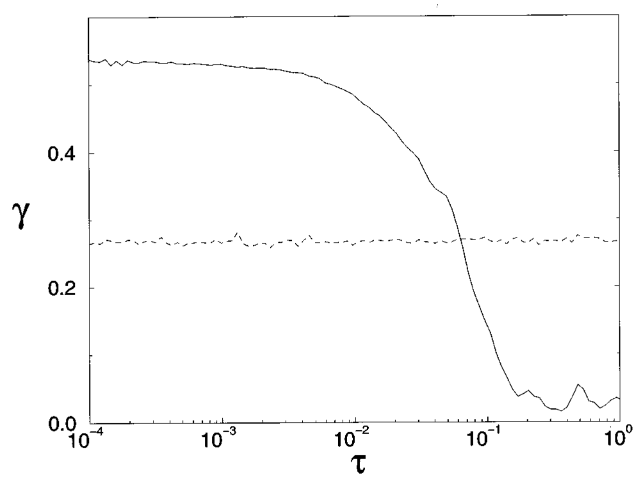

FIG. 8. Minimal Lyapunov exponents $\gamma_{N=12}$ (solid line) and $\gamma_{N=1}$ (dashed line) as a function of τ near QAR for the same model as in Fig. 7. Notice that γ_1 appears to be almost independent of τ, assuming the value corresponding to QAR localization $(\tau\rightarrow 0)$ for all τ.

According to this picture, a saturation of γ_N around an asymptotic value approximately equal to γ_{N*} should occur for arbitrarily low level of chaos (arbitrarily small τ or k), but N^* and γ_{N*} may be very large. We thus expect that, unlike QAR localization, the DL decay can never be faster than exponential, but it should always feature an asymptotic exponential tail. For very small τ, this asymptotic behavior may start only at very small value of the wave function and the decay may look faster than exponential above this value. This is probably what one sees in Fig. 4 for $\tau=10^{-5}$ and $\tau=10^{-3}$. Here an exponential tail may be observed only much below the level of numerical noise $(\sim 10^{-30})$. Because of computational limitations, we were not able to verify the existence of the saturation effect and the associated exponential tail in the DL decay for $\tau<0.1$.

In Fig. 8, we plot γ_N for $N=12$ and $N=1$ as a function of τ near QAR. While γ_{12} changes significantly from its maximal value (near QAR) to its minimal DL value $\approx 2/D$, γ_1 appears to be independent of τ. In fact, the small fluctuations in γ_1 are of the same order of magnitude as the numerical error in this quantity. The case of $N=1$ corresponds to a nearest-neighbor pseudorandom Lloyd model [6,7,9,15]. The independence of the localization length $\xi=1/\gamma$ on τ for such a model has been verified numerically [7,15] for many values of a coupling constant (e.g., k). The potential $V_1(\theta)$ in (48) is a special case of (27),

$$V_1(\theta)=-\frac{2}{k}\arctan[\kappa\cos(\theta)], \tag{51}$$

where $\kappa=-2J_1(k/2)/J_0(k/2)$. For $A=-2/k$ and arbitrary κ_0, the potential (27) leads to nearest-neighbor hopping constants also in the original $M=2$ tight-binding model (39): $W_n=0$, except $W_{\pm 1}=\kappa/2$. This is a two-channel pseudorandom Lloyd model (see some generalizations of this model in Appendix C).

In Ref. [9], the relation $\xi = D/2$ was derived for the KR with the potential (51) by assuming that the pseudorandom disorder can be replaced by a truly random one, giving the usual Lloyd model [19]. The validity of this assumption was verified in Ref. [15]. We now give an alternative and simpler ''proof'' of this relation, based on the *exact* result $\xi_0 = D/2$, which is derived below. The relation $\xi = D/2$ is an immediate consequence of the latter result and the numerically observed independence of γ_1 on τ. The derivation below of $\xi_0 = D/2$ makes use, of course, of the fact that the KR with the potential (51) exhibits QAR (of period 2). As shown in Sec. II, this QAR can occur only if $V(\theta+\pi)+V(\theta)=\text{const}$. This condition is satisfied by the potential (27) only if $\kappa_0 = 0$ with const$=0$, which is precisely the case of (51). As shown below, the relation $\xi = D/2$ is valid also for the equivalent system of the $M=2$ MKR with (51).

To derive the relation $\xi_0 = D/2$, we first perform explicitly the integral in (50) for the potential (51). The final result can be easily expressed in terms of γ in (28):

$$D = 2\langle c^2\rangle\left[\frac{1}{\gamma}+\frac{1}{2}+O(\gamma^2)\right], \tag{52}$$

where we have assumed the strong-chaos regime $\kappa \gg 1$ for (51), corresponding to very small γ in (28). Now, the QAR-localization length ξ_0 for (51) is $\xi_0 = 1/\gamma$ (see Sec. IV) and $\langle c^2\rangle = 1$ for both the KR and the $M=2$ MKR. The relation $\xi_0 = D/2$ for these systems follows then from Eq. (52) in the strong-chaos regime.

From Fig. 7 it appears that γ_2 is also independent of τ for the particular value of k chosen. The case of $N=2$ corresponds to a next-nearest-neighbor pseudorandom model. Such a model was investigated by Brenner and Fishman [15], who found that its localization properties are quite different from those of its truly random counterpart. In particular, the localization length strongly depends on the irrational number (analogous to $\eta = \tau/2\pi$) defining the pseudorandom disorder. Figure 7 suggests, however, that it may be possible to define pseudorandom tight-binding models with $N>1$ neighbors, whose localization length is almost independent of τ. Such models should exhibit localization properties quite similar to those of their truly random counterparts. The existence of these models and related problems are planned to be investigated in a future work [32].

VII. CONCLUSION

In this paper, the problem of quantum localization in 1.5 degrees of freedom (''minimal chaos'') has been studied in a large class of systems exhibiting classical chaotic diffusion, the modulated kicked rotors. These systems feature two basically different kinds of asymptotic exponential localization in angular-momentum space: (a) dynamical localization, which, as in the case of the KR and other systems [6–8,10,11], is the localization exhibited by pseudorandom tight-binding models for irrational values of η (a scaled $\hbar$) [17], and (b) QAR localization, which is the localization occurring for η in the infinitesimal vicinity of points η_0, where the quantum dynamics is exactly periodic (quantum antiresonance).

The existence of QAR localization has been rigorously established in the framework of a self-consistent approach for both period-1 and period-2 QAR (see Sec. III and Appendix A; it seems that QARs of period larger than 2 do not exist for nonintegrable MKRs). This approach leads to the basic equation (23) [or (A11) in Appendix A] for the QE problem in the infinitesimal vicinity of QAR. It follows from this equation that the QE spectrum is pure point for infinitesimal $\eta-\eta_0$ and that the asymptotic QAR-localization length ξ_0 is completely determined from the analytical properties of the potential appearing in the equation: $\xi_0 \leqslant 1/\gamma$, where γ is the smallest distance of a singularity of the potential from the real θ axis. In many interesting cases, such as the potentials (27) and (48), $\xi_0 = 1/\gamma$ exactly.

Being associated with the 1D time-independent Schrödinger equation (23), QAR localization is of an ''integrable'' nature. In fact, if $\eta_0 = 0$ is a QAR point, the Hamiltonian in (23) is precisely the integrable limit ($T\to 0$ at fixed k) of the classical MKR Hamiltonian (12). On the other hand, DL is associated with the difference equations (36) [or (39)] for the pseudorandom tight-binding models. These equations depend not only on the dimensionless kicking parameter k [essentially the only parameter in (23)], but also on η, which determines the pseudorandom disorder. This pseudorandomness, which is absent in the angular-momentum representation of (23), introduces ''nonintegrable'' features in DL.

Thus, while the asymptotic QAR-localization length ξ_0 is completely determined by the analytical properties of the potential in (23), the asymptotic DL length ξ depends, in general, also on k and η. Starting from the integrable case $\eta = \eta_0 = 0$ and increasing the level of classical chaos by increasing η at fixed $k \gg 1$, we find that the sensitivity of ξ to an increase in the analyticity of the potential (exhibited by the hopping constants in the tight-binding model) is gradually reduced. This phenomenon, which has been considered in detail in Sec. VI and is clearly shown in Fig. 7, is a vivid manifestation of classical chaos in quantum dynamics. As soon as one reaches a semiclassical regime ($\eta < 1$) of strong chaotic diffusion ($K = 4\pi\eta k \gg 1$), ξ becomes totally unaffected by any increase in the analyticity of the potential. In this regime, ξ is completely determined by the classical chaotic-diffusion coefficient D, $\xi \approx D/2$.

While QAR localization is basically different from DL, these localizations are expected to look quite similar if η is sufficiently irrational and close to η_0. The numerical results shown in Figs. 2–7 strongly support this expectation. In particular, Figs. 5–7 show that in cases where the QAR-localization length ξ_0 is finite ($\xi_0 \neq 0$) the DL length ξ is well approximated by ξ_0 if η is sufficiently close to η_0. In the case of the potential (51), corresponding to a pseudorandom Lloyd model, ξ appears to be independent of η, so that $\xi = \xi_0$. Using then the fact that the KR with the potential (51) exhibits QAR and $\xi_0 = D/2$ exactly, the relation $\xi = D/2$ for this system follows immediately. Reference [9] presents a much lengthier derivation of this relation, based on the assumption that the pseudorandom disorder can be replaced by a truly random one.

In summary, the presence of QAR in nonintegrable systems is quite useful for studying several interesting aspects of DL, in particular the transition from DL in local-chaos or almost-integrability regimes to DL in the global-chaos regime, and for obtaining exact lower bounds ξ_0 to the asymptotic DL length ξ.

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APPENDIX A: GENERAL QAR OF PERIOD 1 AND PERIOD 2 IN NONINTEGRABLE MKRs

In Sec. III, we have restricted our attention, for simplicity, to a case of period-1 QAR occurring when τ_j in (15) is a multiple of 2π, $\tau_j=2\pi m_j$. Here we shall consider the more general case of τ_j equal to a multiple of π, $\tau_j=m_j\pi$, giving both period-1 and period-2 QARs. We believe, but are unable presently to give an exact proof, that this is actually the *most general* case of QAR in the nonintegrable MKRs (12). We derive the QAR effective Hamiltonian [analogous to (19)] in this general case and obtain an interesting relation [relation (A14) below] between the values of $k_{\rm eff}$ associated with period-1 and period-2 QARs in the $M=3$ MKR.

By repeated application of relation (7), we find that for $\tau_j=m_j\pi$ the evolution operator (14) can be expressed as

$$U=\exp\left[-ik\sum_{j=0}^{M-1}c_jV(\theta+\overline{m_j}\pi)\right]\exp(-i\overline{m_0}\pi\hat{n}^2), \tag{A1}$$

where $\overline{m_j}=\Sigma_{s=j}^{M-1}m_s$. Consider first period-1 QAR. It is clear from (A1) that this QAR may be possible only if $\overline{m_0}$ is even. In this case, one has $U=\exp[-ik\overline{V}(\theta)]$, where $\overline{V}(\theta)$ is the function having Fourier coefficients

$$\overline{v_n}=v_n\sum_{j=0}^{M-1}c_je^{in\overline{m_j}}. \tag{A2}$$

Here v_n are the Fourier coefficients of $V(\theta)$. We denote c_j by $c_{e,j}$ or $c_{o,j}$, depending on whether the corresponding $\overline{m_j}$ is even or odd, respectively. The sum over all $c_{e,j}$ (or all $c_{o,j}$) will be denoted by $c_e=\Sigma c_{e,j}$ (or $c_o=\Sigma c_{o,j}$). Relation (A2) can then be written as

$$\overline{v_n}=v_n[c_e+(-1)^nc_o]. \tag{A3}$$

Now, period-1 QAR, i.e., $U=1$ [without loss of generality, we assume that $\beta=0$ in (2)], implies that $\overline{v_n}=0$ for all n. From relation (A3) we then obtain the following conditions for period-1 QAR, corresponding to two different cases:

$$c_e-c_o=0 \quad \text{if}\ \ V(\theta+\pi)=-V(\theta),$$
$$c_e=c_o=0 \quad \text{otherwise}. \tag{A4}$$

Notice that the second condition [for general $V(\theta)$] leads to the trivial result $c_j=0$ for all j if each of the sets $\{c_{e,j}\}$ and $\{c_{o,j}\}$ contains *only one* element. This may happen only if $M=2$ and m_0 and m_1 are both odd. Conditions (A4) for $\tau_j=m_j\pi$ are a generalization of condition (16) in Sec. III.

If $\overline{m_0}$ is odd, we show that period-2 QAR takes place. We find in this case that $U^2=\exp[-ik\overline{V^{(2)}}(\theta)]$, where $\overline{V^{(2)}}(\theta)$ has Fourier coefficients

$$\overline{v_n^{(2)}}=v_n\sum_{j=0}^{2M-1}c_je^{in\overline{m_j'}}. \tag{A5}$$

Here $\overline{m_j'}=\Sigma_{s=j}^{2M-1}m_s$ and c_j and m_j are "extended" beyond $j=M-1$ by defining, for $j\geqslant M$, $c_j=c_{j-M}$, and $m_j=m_{j-M}$. From the definitions of $\overline{m_j'}$ and $\overline{m_j}$, we see that $\overline{m_j'}=\overline{m_{j-M}}-\overline{m_0}$ for $j\geqslant M$. Using the last relation in (A5), we find that $\overline{v_n^{(2)}}=2v_n\Sigma_{j=0}^{M-1}c_j$ for n even and $\overline{v_n^{(2)}}=0$ for n odd. The conditions for period-2 QAR, i.e., $U^2=1$ or $\overline{v_n^{(2)}}=0$ for all n, are therefore

$$c_j \ \ \text{arbitrary} \quad \text{if}\ \ V(\theta+\pi)=-V(\theta),$$
$$\sum_{j=0}^{M-1}c_j=0 \quad \text{otherwise}. \tag{A6}$$

Thus, if $V(\theta+\pi)=-V(\theta)$ and $\overline{m_0}$ is odd, one has period-2 QAR for *arbitrary* values of c_j. This is a considerable generalization of the period-2 QAR in the KR (see Sec. II), discovered by Izrailev and Shepelyanskii [4]. The second condition in (A6) [for $V(\theta+\pi)\neq-V(\theta)$] is precisely condition (16) in Sec. III.

We now consider small perturbations of τ_j near their QAR values $m_j\pi$. For definiteness, we shall work out in detail here the case of period-1 QAR, i.e., $\overline{m_0}$ even, but we shall show at the end how to extend the results to period-2 QAR. Writing $\tau_j=m_j\pi+\epsilon_j$ in (14) and formally expanding in powers of ϵ_j as in Sec. III, we find, to first order in ϵ_j,

$$U\approx1-\sum_{j=0}^{M-1}\epsilon_j\{i\hat{n}^2-k[2iV_j'(\theta)\hat{n}+V_j''(\theta)]+ik^2V_j'^2(\theta)\}, \tag{A7}$$

where $V_j(\theta)=\Sigma_{s=0}^{j}c_sV(\theta+\overline{m_s}\pi)$. Using (17), the expression in (A7) can be written, to first order in ϵ, as $\exp(-i\epsilon G_1)$, where

$$G_1=[\hat{n}-kV_a'(\theta)]^2+k^2[\Delta V'(\theta)]^2. \tag{A8}$$

Here

$$V_a(\theta)=\sum_{j=0}^{M-1}\frac{t_{j+1}-t_j}{T}V_j(\theta),$$

$$[\Delta V'(\theta)]^2=\sum_{j=0}^{M-1}\frac{t_{j+1}-t_j}{T}V_j'^2(\theta)-V_a'^2(\theta). \tag{A9}$$

The QE problem in the infinitesimal vicinity of QAR is $G_1\psi=g\psi$, and after the gauge transformation

$$\varphi=\exp[-ikV_a(\theta)]\psi, \tag{A10}$$

it reduces to the 1D Schrödinger equation

$$-\frac{d^2\varphi}{d\theta^2}+k^2[\Delta V'(\theta)]^2\varphi=g\varphi. \tag{A11}$$

The analysis of Eq. (A11) is similar to that of Eq. (23) in Sec. III and the conclusions are the same: the QAR-localization length ξ_0 is entirely determined from the analytical properties of the function $[\Delta V'(\theta)]^2$.

The results (A8) and (A11) are valid also in the case of period-2 QAR ($\overline{m}_0$ odd), but all the sums over j [see (A7) and (A9)] run now from $j=0$ to $j=2M-1$ ($t_j=t_{j-M}$ for $j\geqslant M$) and $V_j(\theta)$ is defined as $V_j(\theta)=\Sigma_{s=0}^{j}c_sV(\theta+\overline{m}_s'\pi)$.

As an interesting example, we consider the $M=3$ MKR with $V(\theta+\pi)=-V(\theta)$ and m_j independent of $j=0,1,2$. For $m_j=2$, one has the simple case of period-1 QAR treated in Sec. III, provided, of course, $c_0+c_1+c_2=0$. The value of $k_{\rm eff}$ in (19) can be easily expressed in terms of two independent coefficients, say c_0 and c_1, using (20) $[(t_{j+1}-t_j)/T=1/3$ for all $j]$:

$$k_{\rm eff}=k[2(c_0^2+c_1^2+c_0c_1)/9]^{1/2}. \tag{A12}$$

Suppose now that the period T is halved, $T\rightarrow T/2$, leaving all other parameters (including c_j) unchanged. Then $m_j=1$ for all j and since $\overline{m}_0=3$ (odd) one now has period-2 QAR. The "real-time" period remains then T. Notice also that the condition $c_0+c_1+c_2=0$ is consistent with (A6) (c_j can be chosen arbitrarily in this case). Using $V(\theta+\pi)=-V(\theta)$ in (A9) [with $M\rightarrow 2M$ and $V_j(\theta)=\Sigma_{s=0}^{j}c_sV(\theta+\overline{m}_s'\pi)$], we get

$$k^2[\Delta V'(\theta)]^2=[k_{\rm eff}^{(2)}]^2V'^2(\theta),$$

where

$$k_{\rm eff}^{(2)}=k[5(c_0^2+c_1^2+c_0c_1)/6]^{1/2}. \tag{A13}$$

By comparing (A13) with (A12), we obtain the simple relation, valid for all values of c_0 and c_1,

$$\frac{k_{\rm eff}^{(2)}}{k_{\rm eff}}=(15/4)^{1/2}. \tag{A14}$$

We were not able to discover similar simple relations for other MKR systems.

APPENDIX B: GENERAL QAR IN INTEGRABLE MKR SYSTEMS

It is instructive to study the QAR phenomenon in a "linear" version of the MKR, defined by the general Hamiltonian

$$H=\frac{\tau}{T}L+\hat{k}V(\theta)\sum_{j=0}^{M-1}c_j\Delta_T(t-t_j). \tag{B1}$$

We show that the system (B1) is equivalent, effectively, to the linear KR (9), which is integrable [24] and exactly solvable to a large extent [23]. We then derive general conditions for QAR of arbitrary period p in (2) and show rigorously the existence of exponential localization in the infinitesimal vicinity of QAR.

The evolution operator for (B1), from $t=-0$ to $t=T-0$, can be expressed as

$$U=\prod_{j=0}^{M-1}\exp(-i\tau_j\hat{n})\exp[-ic_jkV(\theta)]$$

$$=\exp(-i\tau\hat{n})\exp[-ik\overline{V(\theta)}], \tag{B2}$$

where $\tau_j=(t_{j+1}-t_j)\tau/T$,

$$\overline{V(\theta)}=\sum_{j=0}^{M-1}c_jV(\theta+\chi_j)], \tag{B3}$$

$\chi_j=\Sigma_{s=0}^{j-1}\tau_j$ for $j\geqslant 1$, and $\chi_0=0$. By comparing the last expression for U in (B2) with Eq. (10), we see that the problem has been reduced, essentially, to that of the linear KR [23] with an "effective" potential $\overline{V(\theta)}$. Thus, from the discussion at the end of Sec. II, it follows that QAR with arbitrary period p occurs precisely at rational values of $\eta=\tau/2\pi=m/p$, provided $\overline{v}_{sp}=0$ for all s. Here $\overline{v}_n$ are the Fourier coefficients of $\overline{V(\theta)}$ in (B3) and are given by

$$\overline{v}_n=v_n\sum_{j=0}^{M-1}c_je^{in\chi_j}, \tag{B4}$$

where v_n are the Fourier coefficients of $V(\theta)$. It is now clear from (B4) that, in contrast with the linear KR case (see Sec. II), the requirement $\overline{v}_{sp}=0$ does not necessarily imply the vanishing of v_{sp}. For simplicity, let us assume in what follows that all τ_j's are equal, $\tau_j=\tau/M$ for all j, so that $\chi_j=j\tau/M$. Then, with $\tau=2\pi m/p$, the condition $\overline{v}_{sp}=0$ is satisfied for all s if

$$\overline{c}_s\equiv\sum_{j=0}^{M-1}c_j\exp(2\pi ijsm/M)=0. \tag{B5}$$

Now, if m and M are relatively prime, the sequence $\overline{c}_s$ in (B5) is, up to some rearrangement for $m\neq 1$, the Fourier transform of c_j ($j,s=0,\ldots,M-1$). Then $\overline{c}_s=0$ necessarily implies that $c_j=0$ and the QAR is trivial for generic potentials $V(\theta)$. If, on the other hand, m and M have a maximal common factor $\overline{m}>1$, the condition $\overline{c}_s=0$ implies only that

$$\sum_{r=0}^{\overline{m}-1}c_{j+rM'}=0 \tag{B6}$$

for $j=0,\ldots,M'-1$, where $M'=M/\overline{m}$. To summarize, the M' equations (B6) are the necessary and sufficient conditions for QAR of period p in the linear MKRs with equal τ_j's ($\tau_j=2\pi m/pM$) and arbitrary potentials $V(\theta)$. These conditions lead to nontrivial results only if $\overline{m}>1$. It should be noticed that the conditions (B6) do not depend on p, i.e., they are the same as those for the fundamental ($p=1$) QAR with $\tau=2\pi m$.

We now consider small perturbations of τ near the conditions (B6). Since the evolution operator (B2) is the same as that of a linear KR with effective potential $\overline{V(\theta)}$, an exact expression of the QE states for irrational η can be immediately written using the results of Ref. [23]:

$$\psi_l(\theta)=\exp[i\varphi_l(\theta)], \tag{B7}$$

where, for all integers l, $\varphi_l(\theta)=l\theta+\phi(\theta)$, and $\phi(\theta)$ is a periodic function with Fourier coefficients

$$\overline{\phi_n}=\frac{k\overline{v_n}}{1-e^{in\tau}} \quad (n\neq 0), \tag{B8}$$

$\overline{v_n}$ being the Fourier coefficients of $\overline{V(\theta)}$, and $\overline{\phi_0}$ is arbitrary. The corresponding quasienergy is given by $\omega_l=\overline{v_0}+l\tau$. Now, in the ordinary case of the linear KR, the result analogous to (B8) [with $\overline{v_n}$ replaced by the given coefficients v_n of $V(\theta)$] is clearly not defined for $n=sp$ and $\eta=m/p$, i.e., at quantum resonance. In our case, however, with the coefficients $\overline{v_n}$ given by (B4), it is easy to show that the expression (B8) for $n=sp$ is well defined in the QAR limit $\eta\rightarrow m/p$, the limit being taken on some sequence of irrational η's converging to m/p. In fact, writing $\tau=(2\pi m+\epsilon)/p$ and using the conditions (B6), we find from (B4) (with $\chi_j=j\tau/M$) that

$$\overline{v_{sp}}=v_{sp}\sum_{j=0}^{M'-1}e^{2\pi ijsm'/M'}\sum_{r=0}^{\overline{m}-1}c_{j+rM'}(e^{is(j+rM')\epsilon/M}-1). \tag{B9}$$

After substituting (B9) into (B8) and using (B6), we obtain

$$\lim_{\eta\rightarrow m/p}\overline{\phi_{sp}}=-\frac{kv_{sp}}{\overline{m}}\sum_{j=0}^{M'-1}e^{2\pi ijsm'/M'}\sum_{r=0}^{\overline{m}-1}rc_{j+rM'}. \tag{B10}$$

Relation (B10) shows that the QE states are well defined in the infinitesimal vicinity of the QAR of arbitrary period p. We see from (B8) and (B10) that these QE states are localized in L space with the same localization length of the (analytic) potential $\overline{V(\theta)}$. In accordance with this, the QE spectrum of U^p in the limit of infinitesimal ϵ is pure point, $\omega_l \bmod 2\pi=\epsilon l$.

These rigorous results, which have no analog in the ordinary case of the linear KR [23], can be derived by an alternative approach, similar to that used for the nonintegrable MKRs in Sec. III. The basic evolution operator U^p can be expressed in the form (11) with $V(\theta)$ replaced by $\overline{V(\theta)}$. Then, for $p\tau=2\pi m+\epsilon$, one easily finds, using (B6), that U^p is given, to first order in ϵ, by $\exp(-i\epsilon G_1)$, where

$$G_1=\hat{n}-\phi_0'(\theta), \tag{B11}$$

with $\phi_0(\theta)=\lim_{\epsilon\rightarrow 0}\phi(\theta)$. In the case of the fundamental QAR ($p=1$), with $\tau_j=2\pi m_j$ in (B2) as in Sec. III, we find that $\phi_0(\theta)=kdV(\theta)$. Thus the operator (B11) may be considered as the linear version of (19). Its eigenvalues are simply all the integers l and its eigenstates are given by (B7) with $\phi(\theta)$ replaced by $\phi_0(\theta)$. We then see that in the limit $\epsilon\rightarrow 0$ the QE spectrum and eigenstates of the approximate evolution operator $U^p=\exp(-i\epsilon G_1)$ agree precisely with the rigorous ones obtained above. This may be evidence that the results obtained by the self-consistent approach in Sec. III are, in fact, rigorous.

APPENDIX C: NEARLY SINGULAR CASES OF MULTICHANNEL TIGHT-BINDING MODELS

Clearly, all the hopping constants $\overline{W_{s,n}}$, $s\neq 0$, in the singular case [Eq. (38)] have the same range in n (i.e., the range of W_n). It is thus interesting to consider nearly singular cases of (36) ($M>2$) for which the hopping constants $W_{0,n}$ within a channel are nonzero, while $W_{s,n}$ for $s\neq 0$ still have approximately the same range in n. The simplest possible case where this may happen is when c_j takes only the values $0,\pm 1,\pm 2$. In this case, the Fourier transform $W_s(\theta)$ of $W_{s,n}$ assumes the relatively simple form

$$\overline{W_s}(\theta)=c^{(1)}(s)W(\theta)+c^{(2)}(s)\frac{W(\theta)}{1-W^2(\theta)}, \tag{C1}$$

where $c^{(1)}(s)$ and $c^{(2)}(s)$ are, respectively, the contributions of the terms with $c_j=\pm 1$ and $c_j=\pm 2$ to $c(s)$ in (38). Using $c^{(1)}(0)+c^{(2)}(0)=0$ [relation (16)], we find that

$$\overline{W_0}(\theta)=-c^{(1)}(0)\frac{W^3(\theta)}{1-W^2(\theta)}. \tag{C2}$$

As an example, consider the case of $W(\theta)=\kappa\cos(\theta)-\kappa_0$, which corresponds to the potential (27) ($A=-2/k$) and gives a nearest-neighbor (Lloyd) model (39) for $M=2$, i.e., $W_n=0$, except $W_{\pm 1}=\kappa/2$ (such a model is studied in Sec. VI). We see from (C1) and (C2) that there are two interesting limits. If $\kappa_0\gg\max(\kappa,1)$, all the hopping constants $\overline{W_{s,n}}$ are approximately nearest neighbor in n, and $E=c^{(1)}(0)\kappa_0$. If, on the other hand, $\max(\kappa,\kappa_0)\ll 1$, $W_{s,n}$ are approximately nearest neighbor for $s\neq 0$, while $W_{0,n}$ are approximately next to next nearest neighbor and $E\approx c^{(1)}(0)\kappa_0^3$. The localization length $\xi=1/\gamma$, however, depends only on $W(\theta)$, and it is always given exactly by relation (28).

It should be noticed, however, that the choice $W(\theta)=\kappa\cos(\theta)-\kappa_0$ is not a good one for obtaining a nearly nearest-neighbor model in the strong-chaos regime ($D\gg 1$). This is because Eqs. (52) and (28) imply that $|\kappa|,|\kappa/\kappa_0|\gg 1$ in this regime, so that $W(\theta)=\kappa\cos(\theta)-\kappa_0=1$ for some θ, leading to a singularity in Eqs. (C1) and (C2). Let us choose instead

$$\frac{2W(\theta)}{1-W^2(\theta)}=-\tan[kV(\theta)]=\kappa\cos(\theta)-\kappa_0, \tag{C3}$$

i.e.,

$$W(\theta)=\frac{\kappa\cos(\theta)-\kappa_0}{1+\{1+[\kappa\cos(\theta)-\kappa_0]^2\}^{1/2}}. \tag{C4}$$

It is clear from (C4) that in the strong-chaos regime $|W(\theta)|\approx 1$ for almost all θ. Equations (C1)–(C3) imply then that the hopping constants $W_{s,n}$ are all nearest neighbor in n to high accuracy, i.e., $W_{s,n}\approx 0$ except $W_{s,\pm 1}\approx c^{(2)}(s)\kappa/4$. The corresponding tight-binding model (36) may thus be naturally viewed as a multichannel pseudorandom Lloyd model for the strong-chaos regime.

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[17] The term ''dynamical localization'' is mainly used in a situation of global chaotic diffusion, where the extended classical motion is in sharp contrast with the localized quantum motion (see, e.g., Ref. [2]). However, the localization in pseudorandom tight-binding models takes place, of course, also in local-chaos or almost-integrability regimes, where the classical motion is localized as the quantum one. As shown in Refs. [6,15], the fundamental reason for this localization is the pseudorandom disorder, produced by a sufficiently irrational η. The nature of chaos (local or global) and its strength are expected to affect only several features of this localization, such as the localization length. We shall therefore use the term ''dynamical localization'' in this paper also in local-chaos or almost-integrability regimes.
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