Dynamical Localization near Quantum Antiresonance: Exact Results and a Solvable Case

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Dynamical localization in general two-sided kicked rotors, which are classically *nonintegrable,* is shown to occur in the immediate vicinity of quantum antiresonance (periodic recurrences). A complete and exact solution of the quasienergy eigenvalue problem is obtained for the standard potential. Numerical evidence is given that this solution is an excellent approximation to the quantum dynamics and quasienergy states even not very close to antiresonance. The dynamical problem is mapped into a tight-binding model of a two-channel strip with pseudorandom disorder. One then has strong evidence for Anderson localization in this model near antiresonance.

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The quantum dynamics of classically nonintegrable systems has been extensively investigated during the last two decades [1]. A favorite class of model systems has been the periodically kicked rotors (KR) [2–9], which, while relatively simple, already exhibit the variety and complexity of classical structures and motions (regular and chaotic) present in generic Hamiltonian systems. In particular, by increasing the nonintegrability (kicking) parameter in KR systems, one observes the typical gradual destruction of isolating Kol'mogorov-Arnol'd-Moser tori in phase space. This leads to the well-known transition from bounded to global chaos, featuring unbounded diffusion in the angular momentum L [10].

As strongly indicated by extensive numerical and analytical studies, this diffusion is generically suppressed in the quantum case [2,3] and is replaced by bounded variation of $\langle L^2 \rangle$ with quasiperiodic recurrences. This phenomenon can be attributed to a pure-point quasienergy (QE) spectrum (i.e., the spectrum of the one-period evolution operator) [3]. An understanding of the generic occurrence of such a spectrum was achieved by showing that the QE eigenvalue problem is formally equivalent, in the L representation, to a 1D tight-binding model [4]. For generic (irrational) values of a dimensionless $\hbar$, denoted here by η, the on-site potential in this model is pseudorandom. Several arguments have been given [4,11] that the effect of such a potential is quite similar to that of a truly random one, which causes Anderson exponential localization of all the electronic eigenstates [12]. Assuming this, the QE states feature a similar "dynamical" localization in L space, and the QE spectrum is pure point.

In order to get a better understanding of the nature of quantum suppression of diffusion and its relation to Anderson localization in disordered solids, rigorous results and/or exactly solvable models are most desired. The existence of a pure-point QE spectrum was proven in some special variants of the original KR system [13]. However, in the case that these variants have a classical counterpart, the nonintegrability parameter assumed in the proof is usually too small for unbounded diffusion to take place. Exact solutions of the QE eigenvalue problem for the KR can be found for rational values of η, the so-called quantum resonances [14]. In this nongeneric case, however, the corresponding on-site potential is periodic, and the QE states are extended Bloch states leading to a ballistic (quadratic) increase of $\langle L^2 \rangle$ with time. An exactly solvable variant of the KR is obtained after replacing the kinetic energy of the rotor by a linear function of L [15]. The resulting system can be mapped into a tight-binding model whose on-site potential is incommensurate with the lattice for irrational η. While such potential is not even pseudorandom (it is quasiperiodic) [11], it can lead to the same exponential localization of eigenstates as in random Anderson models. Unfortunately, however, this system is classically integrable [16], so that the localization has nothing to do with the suppression of chaotic diffusion, but it is rather associated with regular motion.

In this Letter, we show that dynamical localization occurs in a class of *nonintegrable* systems, for arbitrary values of a nonintegrability parameter and for η in the immediate vicinity of values η_0 associated with a distinctive quantum phenomenon. For a standard system in this class, we obtain a complete and exact solution of the QE eigenvalue problem in terms of transcendental functions. We give strong numerical evidence that this solution is an excellent approximation to the quantum dynamics and QE states even for $|\eta - \eta_0|$ not very small. The dynamical problem is mapped into a tight-binding model of a two-channel strip [17] with pseudorandom disorder. One then has strong evidence for Anderson localization in this model for small values of $|\eta - \eta_0|$.

The class of systems considered are the two-sided kicked rotors (TKRs) [9], defined by the Hamiltonian

$$H = \frac{L^2}{2I} + \hat{k} V(\theta) \sum_{s=-\infty}^{\infty} (-1)^s \delta\left(t - \frac{sT}{2}\right), \qquad (1)$$

where I is the moment of inertia, $\hat{k}$ is the kicking parameter, T is the time period, and $V(\theta)$ is a general periodic and analytic function of the angle θ. Two-sided kicking perturbations such as in (1) were considered in several physical contexts [18] as approximations of sinusoidal driving

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potentials corresponding to ac electromagnetic fields. By increasing $\hat{k}$ in the classical TKR, one observes the typical transition from bounded to global chaos [9], as in the KR case. The quantum dynamics is governed, as usual, by the evolution operator U in one period, e.g., from $t = -0$ to $t = T - 0$,

$$U = e^{-i\tau\hat{n}^2} e^{ikV(\theta)} e^{-i\tau\hat{n}^2} e^{-ikV(\theta)}, \tag{2}$$

where $\hat{n} \equiv L/\hbar = -id/d\theta$, $\tau \equiv \hbar T/4I$, and $k \equiv \hat{k}/\hbar$. Evidence for dynamical localization is provided by a localized "steady-state" probability distribution f_n over the angular momentum $n\hbar$ [5–7]. We observed numerically that this indeed occurs in the TKR; see an example in Fig. 1. As we shall show below, this localization is to be expected for generic irrational values of $\eta \equiv \tau/2\pi$, as in the KR case [4–7]. For rational values of $\eta = m/p$, where m and p are relatively prime integers and $p > 1$, one can show [19] that the QE spectrum of (2) consists of p bands of Bloch states, leading to a ballistic increase of $\langle L^2 \rangle$ with time. This is again analogous to the KR case [14].

But a most distinctive feature of the quantum TKR is that U becomes the identity operator for $\eta = m$, an integer [since the operator $\exp(-i\tau\hat{n}^2)$ in (2) is clearly the identity in this case]. This implies *exactly* periodic recurrences (with period 1) of an arbitrary wave packet [9], a phenomenon diametrically opposite to that occurring in the KR for $\eta = m$, namely, the fundamental quantum resonance [14]. We shall therefore refer to this phenomenon as quantum antiresonance (QAR). In a broad sense, the QAR appears to arise in several quantum-dynamical problems [19,20]. Since $U = 1$, the QE spectrum consists just of a single, infinitely degenerate level. The natural question is then precisely how this infinite degeneracy is removed by slightly perturbing η near $\eta = m$. As we now show, this degeneracy is removed quite abruptly: The QE spectrum of U for $\tau = 2\pi\eta = 2\pi m + \epsilon$ and infinitesimal $\epsilon \neq 0$ is pure point, and the QE states are exponentially localized in L space. For the standard potential $V(\theta) = \cos(\theta)$, the QE eigenvalue problem will be solved exactly.

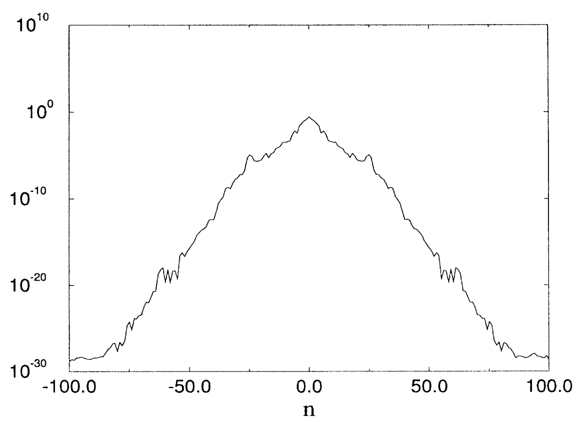

FIG. 1. Example of a localized steady-state probability distribution f_n in the TKR for the standard potential $V(\theta) = \cos(\theta)$, and for $k = 2.0$ and $\tau = 1.0$. Here f_n is obtained from an initial wave packet $|\phi_0\rangle = |n = 0\rangle$ by averaging $|\langle n|U^s|\phi_0\rangle|^2$ over 20 000 iterations in the interval $190\,000 < s < 210\,000$.

Using the operator identity

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots, \tag{3}$$

and expanding the operator $\exp(-i\tau\hat{n}^2) = \exp(-i\epsilon\hat{n}^2)$ in powers of ϵ, it is easy to see that U in (2) can be formally written as $\exp(-iG)$, $G = \sum_{j=1}^{\infty} \epsilon^j G_j$ [21]. Here the Hermitian operators G_j are polynomials in $\hat{n}$ and derivatives of $V(\theta)$ of order not larger than $2j$. For analytic $V(\theta)$, these operators are all well defined. It is now clear that in the limit of infinitesimal $\epsilon \neq 0$, the QE states are precisely the eigenstates of the leading operator $G_1 = G/\epsilon$ [21]. A tedious but straightforward calculation gives

$$G_1 = 2[\hat{n} - (k/2)V'(\theta)]^2 + (k^2/2)\,V'^2(\theta), \tag{4}$$

where $V'(\theta) = dV(\theta)/d\theta$. If the eigenvalue problem for G_1 is $G_1\psi = g\psi$, we perform the gauge transformation

$$\varphi = \exp[-ikV(\theta)/2]\psi, \tag{5}$$

and obtain for φ, using (4), the eigenvalue equation

$$-\frac{d^2\varphi}{d\theta^2} + \frac{k^2}{4}V'^2(\theta)\varphi = \frac{g}{2}\varphi. \tag{6}$$

The problem has thus been reduced to that of a Schrödinger equation with a periodic potential. The spectrum g then has a band structure, but because of the periodic boundary condition $\varphi(2\pi) = \varphi(0)$, only the level with zero quasimomentum is picked out from each band. This gives, in general, a point spectrum. Now, being the solution of the linear differential equation (6), $\varphi(\theta)$ is analytic at least in the domain of analyticity of $V'(\theta)$ [22]. Let γ be the smallest distance of a singularity of $V'(\theta)$ from the real θ axis. Then the Fourier-series expansion of $\varphi(\theta)$ will converge at least within an infinite horizontal strip of width 2γ, symmetrically positioned around the real θ axis [22]. It follows that the Fourier coefficients of φ and ψ in (5) decay at least as $\exp(-\gamma|n|)$, and the localization length is not larger than $1/\gamma$ (see example later).

For the standard potential $V(\theta) = \cos(\theta)$, Eq. (6) becomes the *Mathieu equation* [22,23]

$$y'' + [a - 2q\cos(2\theta)]y = 0, \tag{7}$$

where $y = \varphi$, $a = g/2 - k^2/8$, and $q = -k^2/16$. The problem is then exactly solved in terms of the periodic Mathieu functions $y = ce_r(\theta,q)$ (symmetric) and $y = se_r(\theta,q)$ (antisymmetric), with corresponding eigenvalues $a = a_r(q)$ and $a = b_r(q)$. Explicit expressions for these functions and eigenvalues, as well as a detailed discussion of their properties, can be found in Refs. [22,23]. From Eq. (5) the Fourier coefficients ψ_n and y_n of ψ and $y = \varphi$, respectively, are related by $\psi_n = \sum_j i^j J_j(k/2) y_{n-j}$, where

$J_j(k/2)$ is a Bessel function. Since the dominant decay rate of both J_n and y_n with n is like n^{-n} [23], this is also the dominant decay rate of ψ_n. This strong localization in L space, faster than exponential, could be expected from the fact that $V(\theta) = \cos(\theta)$ is an entire function, so that the localization length $1/\gamma = 0$.

When $\epsilon/2\pi$ is not infinitesimal but sufficiently small and irrational, we expect this exact solution to be a good approximation to QE states which are localized within an interval of length $1/\sqrt{\epsilon}$ around $n = 0$. This is because the derivation above was based on an expansion in powers of $\epsilon\hat{n}^2$. To check this, we investigated numerically the quantum dynamics of a wave packet initially equal to $|n = 0\rangle$. A basis of up to 512 angular-momentum states around $n = 0$ was used. In Fig. 2, we plot the kinetic-energy expectation value, $E_0 = \langle L^2/2I\rangle$, as a function of the "real time" $t = \epsilon s$ (s is the number of applications of U), for several values of ϵ. We observe that all the data fall quite accurately on the same curve, even for values of ϵ as large as $\epsilon = 0.33$. This is evidence that, even for ϵ not very small, the quantum dynamics is very well described by the evolution operator $\bar{U}^s = \exp(-i\epsilon s G_1) = \exp(-itG_1)$, generated by the operator G_1 upon which the exact solution is based (see above). In Fig. 3, we plot the Fourier transform $E_0(\nu)$ of $E_0(t)$ for $\epsilon = 0.05$ and several values of k. The positions of the various peaks in $E_0(\nu)$ must correspond the values of $\epsilon\nu$ equal to the spacings between QE levels. A comparison with the level spacings corresponding to eigenvalues of the Mathieu equation shows excellent agreement. This is strong additional evidence that the quantum dynamics and QE states near QAR are very well described by the approximate evolution operator $\bar{U} = \exp(-i\epsilon G_1)$ with generator G_1.

The general dynamical problem will now be mapped into a tight-binding model. Let $u_j^{\pm}(\theta)$, $j = 0, 1$, denote a QE state with quasienergy ω at time $t = jT/2 \pm 0$. The

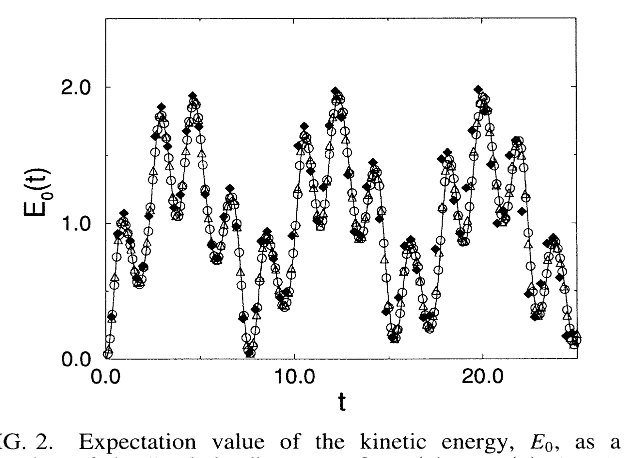

FIG. 2. Expectation value of the kinetic energy, E_0, as a function of the "real time" $t = \epsilon s$, for $V(\theta) = \cos(\theta)$, $k = 2$, and several values of ϵ. The continuous curve corresponds to $\epsilon = 0.013$, the circles to $\epsilon = 0.12$, the triangles to $\epsilon = 0.17$, and the filled diamonds to $\epsilon = 0.33$.

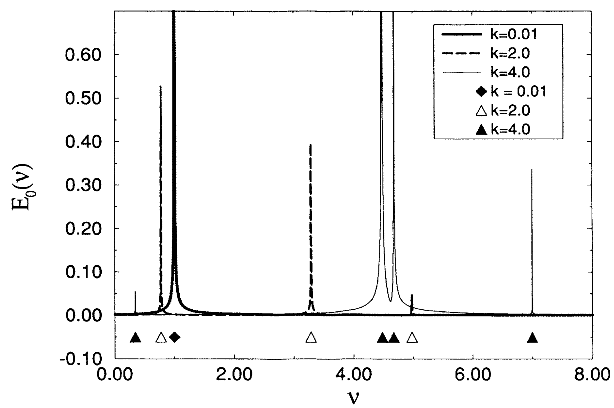

FIG. 3. Fourier transform $E_0(\nu)$ of $E_0(t)$ for $V(\theta) = \cos(\theta)$, $\epsilon = 0.05$, and several values of k (see legend). The symbols at the bottom are the theoretical predictions for the peak positions, based on the eigenvalues of the Mathieu equation (7). The single peak for $k = 0.01$ has been rescaled by a factor of 50 000 for visibility.

following relations hold:

$$u_j^{+}(\theta) = \exp[-i(-1)^j kV(\theta)]u_j^{-}(\theta), \qquad (8)$$

$$u_{1,n}^{-} = e^{-i\tau n^2}u_{0,n}^{+}, \quad u_{0,n}^{-} = e^{i(\omega-\tau n^2)}u_{1,n}^{+}, \qquad (9)$$

where $u_{j,n}^{\pm}$ is the L representation of $u_j^{\pm}(\theta)$. We define, in some analogy with Ref. [4],

$$u_j(\theta) = e^{ij\omega/2}\frac{u_j^{+}(\theta) + u_j^{-}(\theta)}{2}, \qquad e^{-ikV(\theta)} = \frac{1 + iW(\theta)}{1 - iW(\theta)}, \qquad (10)$$

so that $W(\theta) = -\tan[kV(\theta)/2]$. Simple manipulations of Eqs. (8), (9), and (10) yield then the system of equations

$$T_n u_{0,n} + S_n u_{1,n} + \sum_{r\neq 0} W_{n-r}u_{0,r} = Eu_{0,n}, \qquad -T_n u_{1,n} - S_n u_{0,n} + \sum_{r\neq 0} W_{n-r}u_{1,r} = Eu_{1,n}, \qquad (11)$$

where $T_n = \cot(\alpha_n)$, $S_n = -1/\sin(\alpha_n)$, $\alpha_n = \tau n^2 - \omega/2$, W_n are the Fourier coefficients of $W(\theta)$, and $E = -W_0$. Equations (11) describe a tight-binding model of a two-channel strip [17]. The on-site potential and hopping constants within each channel are, respectively, T_n and W_n, while the coupling constants between the channels are S_n. For irrational $\eta = \tau/2\pi$, both T_n and S_n are pseudorandom sequences [4,11] which, by arguments similar to those used in the KR case [4], may lead to Anderson-like localization of the eigenstates of (11) (see Fig. 1). In the neighborhood of the QAR, i.e., for small values of $\epsilon = \tau \bmod 2\pi$, the quasienergy ω is given approximately by $\omega = \epsilon g$, where g is an eigenvalue of the leading operator G_1 (see above). Then $\alpha_n \bmod 2\pi \approx \epsilon(n^2 - g/2)$, so that both T_n and S_n can be much larger than W_n for n not too large. Moreover, when approaching the limit of infinitesimal ϵ, the pseudorandomness of T_n and S_n is guaranteed by choosing $\epsilon/2\pi$ in a sequence of "strong" irrationals $\epsilon_l/2\pi$, $l = 1, 2, \ldots$, converging to 0 [e.g., $\epsilon_l = 2\pi/(l + \lambda)$, where λ is the golden mean]. It is then very likely that for $\epsilon = \epsilon_l$ the model (11) will have eigenstates exponentially localized near $n = 0$. This is also supported by the numerical results presented above, which strongly indicate that the solution of Eq. (6), in particular the explicit exact solution of the Mathieu equation (7), is an excellent approximation to these eigenstates even for l not too large. The accuracy of this approximation can be arbitrarily increased by increasing l. Using Eqs. (5), (8), and (10), it is easy to show that the accurate relation expected between the solutions of Eqs. (6) and (11) for small ϵ reads simply $u_0(\theta) \approx \cos[kV(\theta)/2]\varphi(\theta)$. Besides the exactly solvable case of $V(\theta) = \cos(\theta)$, another interesting case is that of $V(\theta) = -(2/k)\arctan[\kappa\cos(\theta) - E]$, for which only nearest-neighbor hopping appears in (11) ($W_n = 0$ except of $W_{\pm 1} = \kappa$). In this case, the localization length $1/\gamma$ is determined from $\gamma = |\mathrm{Im}(\theta_0)|$, where θ_0 is a pole of $V'(\theta)$, satisfying the equation $\kappa\cos(\theta_0) - E = i$. It is easily verified that $1/\gamma$ is precisely the localization length for the Lloyd model [12] and the linear model [15].

In conclusion, we have shown the occurrence of dynamical localization in a class of nonintegrable systems (the TKRs with arbitrary analytic potential), exhibiting unbounded chaotic diffusion. This localization takes place in the infinitesimal neighborhood of QAR, where a pure-point QE spectrum replaces the infinitely degenerate level at QAR. The QE problem has been expressed as a Schrödinger equation with a periodic potential. From the analytic properties of this potential one can then determine exactly, apparently for the first time, the localization length in a nonintegrable system. Moreover, in the case of the standard potential, the entire QE problem can be solved exactly in terms of Mathieu and Bessel functions. Strong numerical evidence has been given that this solution is an excellent approximation to the quantum dynamics and QE states even not very close to QAR. By mapping the dynamical problem into a tight-binding model of a two-channel strip with pseudorandom disorder, one has then strong evidence for Anderson localization in this model near QAR.

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[1] See, for example, M. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, Berlin, 1990), and references therein.

[2] G. Casati, B. V. Chirikov, F. M. Izrailev, and J. Ford, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, edited by G. Casati and J. Ford, Lecture Noted in Physics Vol. 93 (Springer, Berlin, 1979); B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, Sov. Sci. Rev. **2C**, 209 (1981); F. M. Izrailev, Phys. Rep. **196**, 299 (1990).

[3] T. Hogg and B. A. Huberman, Phys. Rev. Lett. **48**, 711 (1982).

[4] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982); Phys. Rev. A **29**, 1639 (1984).

[5] D. L. Shepelyansky, Physica (Amsterdam) **28D**, 103 (1987).

[6] E. Doron and S. Fishman, Phys. Rev. Lett. **60**, 867 (1988).

[7] G. Casati, I. Guarneri, and I. Shepelyansky, Phys. Rev. Lett. **62**, 345 (1989).

[8] A. Altland, Phys. Rev. Lett. **71**, 69 (1993).

[9] E. Eisenberg and N. Shnerb, Phys. Rev. E **49**, R941 (1994).

[10] B. V. Chirikov, Phys. Rep. **52**, 263 (1979); J. M. Greene, J. Math. Phys. **20**, 1183 (1979); I. Dana and S. Fishman, Physica (Amsterdam) **17D**, 63 (1985); I. Dana, N. W. Murray, and I. C. Percival, Phys. Rev. Lett. **62**, 233 (1989).

[11] M. Griniasty and S. Fishman, Phys. Rev. Lett. **60**, 1334 (1988); N. Brenner and S. Fishman, Nonlinearity **4**, 211 (1992).

[12] N. F. Mott and W. D. Twose, Adv. Phys. **10**, 107 (1961); K. Ishii, Prog. Theor. Phys. Suppl. **53**, 77 (1973).

[13] J. Bellissard, in *Trends and Developments in the Eighties*, edited by S. Albeverio and Ph. Blanchard (World Scientific, Singapore, 1985); M. Combescure, J. Stat. Phys. **59**, 679 (1990).

[14] F. M. Izrailev and D. L. Shepelyanskii, Theor. Math. Phys. **43**, 553 (1980).

[15] D. R. Grempel, S. Fishman, and R. E. Prange, Phys. Rev. Lett. **49**, 833 (1982); R. E. Prange, D. R. Grempel, and S. Fishman, Phys. Rev. B **29**, 6500 (1984); S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. B **29**, 4272 (1984).

[16] M. V. Berry, Physica (Amsterdam) **10D**, 369 (1984).

[17] B. Derrida, K. Mecheri, and J. L. Pichard, J. Phys. (Paris) **48**, 733 (1987), and references therein.

[18] Y. Gu and J. M. Yuan, Phys. Rev. A **36**, 3788 (1987); G. Casati, I. Guarneri, and G. Mantica (to be published).

[19] I. Dana, E. Eisenberg, and N. Shnerb (to be published).

[20] I. Dana, Phys. Rev. Lett. **73**, 1609 (1994).

[21] We assume that this formal expansion is valid, at least in the sense of an asymptotic series, in some finite neighborhood of $\epsilon = 0$.

[22] E. L. Ince, *Ordinary Differential Equations* (Dover, London, 1956); E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (University Press, Cambridge, 1952), 4th ed.

[23] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).