Critical behavior with axially correlated random bonds

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Critical properties are studied in systems with quenched bond disorder that is correlated along d_1 of d dimensions. A renormalization-group scheme (based on the Migdal-Kadanoff method) which follows the full distribution of the random bonds and which gives correctly the modified Harris criterion $\phi=\alpha+d_1\nu$ is used. For $d_1<d-1$, we find fixed distributions at finite temperatures, yielding new "random" exponents for various q-state Potts models. For $d_1=d-1$ there is no long-range order if there is a finite weight to zero coupling. Otherwise, we find a novel zero-temperature fixed distribution, for which all the moments diverge to infinity with finite ratios among them. This fixed distribution has a magnetic eigenvalue equal to d, indicating a first-order transition in the magnetization and possible related essential singularities. Thus, by analogy, the possibility of a magnetization jump is raised for the McCoy-Wu transition on a square lattice. The results for $d_1=1$ are relevant to random quantum systems.

I. INTRODUCTION

The critical properties of systems with *quenched random interactions* have been the subject of much interest in recent years.[1] It is generally accepted that random exchange interactions, with short-range correlations, have a rather small effect (at least far above the percolation threshold). Harris[2] used a heuristic argument to show that the critical behavior is unchanged if $\alpha=2-d\nu<0$, where α is the specific-heat exponent, ν the correlation-length exponent, and d the dimensionality of the "pure" (nonrandom) system. Momentum-space renormalization-group[3-7] (RG), general scaling,[6,7] and position-space RG arguments[8,9] were later introduced to show that α is equal to the crossover exponent of the width of the distribution of the random exchange interactions. When $\alpha>0$, one expects a crossover to a new behavior, characterized by a RG flow to a finite-width *fixed distribution* of the exchange variables. Such a fixed distribution was indeed found by both momentum space[3-7] and position space RG calculations.[8,9]

A different type of randomness arises in the case of *quantum-mechanical spin models* at zero temperature.[10] A well-studied example concerns the Ising model in a transverse field, with the Hamiltonian

$$\mathcal{H}=-\sum_i \Gamma_i\sigma_i^x-\sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z\ . \tag{1.1}$$

In the pure case, this model exhibits a phase transition at zero temperature $T=0$, as a function of the transverse field Γ. The properties of this transition in d dimensions may be found by looking at a $(d+1)$-dimensional classical ($\Gamma=0$) Ising model, with the coupling in the additional "time" direction related to Γ.[11-15] Approximate RG calculations for the random case, in $d=1$, found a "random" fixed distribution of J's and Γ's, with a negative specific-heat exponent.[10] An alternative approach would be to map the random quantum case onto a $(d+1)$-dimensional classical model. Clearly, the randomness in the corresponding coupling constants will occur only in the d-dimensional "space" coordinates, and not along the additional time direction. The random interactions are fully correlated along this direction, having the same value along one-dimensional "rods." These statements can be easily generalized to any quantum problem.[13,15] Similar rodlike random interactions may arise due to extended linear (or planar) defects in crystals. All of these serve as physical motivations for studying correlated impurities.

In this paper we consider the general case, in which the random interactions are correlated along d_1 of the d spatial dimensions. In the first exact analysis of such a case (the Ising model with $d=2$, $d_1=1$), McCoy and Wu[16] found a so-called "smeared" transition which exhibited an essential singularity (i.e., $\alpha=-\infty$) in the specific heat at the transition. However, our present work suggests a possible discontinuous onset of the magnetization at this transition. One can generalize all the arguments listed above for the Harris criterion and show that the crossover exponent for such rodlike randomness is

$$\phi=\alpha+d_1\nu\ . \tag{1.2}$$

Lubensky,[3] who first derived Eq. (1.2) using momentum space RG, found a "runaway" of the RG flows in $d=4-\epsilon$ dimensions, which he interpreted as indicating a smeared transition. More recently, several authors employed a double expansion in ϵ and in d_1 to find stable random fixed points.[17-19] However, similar expansion of the Potts model[20] in $\epsilon=6-d$ and in d_1-2 yielded no stable random fixed points[21] and was interpreted as yielding a smeared transition. We note again that these runaways could also signify first-order transitions.

Equation (1.2) shows that the exponent ϕ is usually positive and large, and therefore the RG recursion relations are expected to flow far away from the pure fixed point. Such drastic changes in the critical behavior will hopefully be easily observable in real (or computer) experiments. However, they are usually difficult to find using a perturbative expansion, as used in the momentum-space RG calculations.[3,17–19,21] In the present paper we therefore employ the alternative approach of *position-space* RG. Following the ideas of Ref. 9, we generate the recursion relations for the *full (correlated) distribution of coupling constants,* and search for *fixed distributions.*

As we show below (Sec. II), there is a variety of ways to introduce correlated randomness. In the earlier work, the correlations were always along the same d_1 axes, and this resulted in anisotropic correlations, e.g., different correlation lengths along the d_1 correlated axes, $\xi_{\|}\sim t^{\nu_{\|}}$ and along the remaining axes, $\xi_{\perp}\sim t^{\nu_{\perp}}$. Permutations among axes will generate an isotropic behavior.

Specifically, we study the correlated random q-state Potts model, using the Migdal-Kadanoff position space approach.[22,23] This approach, which is exact on hierarchical lattices,[24] has the advantage that Eq. (1.2) is exactly obeyed (with the appropriate approximate exponents). The exponent ϕ can be continuously varied[24,25] by changing the number of states, q. A discussion of Eq. (1.2) is given in Sec. III. Our detailed recursion relations are then described in Secs. IV and V.

Unlike the perturbative calculations,[21] we do find random fixed distributions for all q's when $d=3$, $d_1=1$, and $\phi>0$. These are described in Sec. VI. In contrast, we find a different behavior for the McCoy-Wu case, $d=2$, $d_1=1$, as well as for $d=3$, $d_1=2$. We believe this behavior to apply whenever $d_1=d-1$, in which case the system will break into finite one-dimensional rods for any infinitesimal concentration of zero bonds. If this concentration is zero, we find a flow to a new random fixed distribution at $T=0$, with a magnetic exponent equal to d. This is interpreted as a first-order transition.[26,27] The essential singularity found by McCoy and Wu[16] may thus be related to that encountered (as function of the magnetic field) at the first-order transition which occurs at the coexistence curve.[28] Conversely, the McCoy-Wu transition could have a magnetization jump from zero. These results are discussed in Sec. VII, and our conclusions are given in Sec. VIII.

II. THE MODEL

The quenched disorder that we consider in this paper is axially correlated bond disorder on hypercubic lattices, where the bonds are independent random variables in $d-d_1$ dimensions, d being the lattice dimensionality, and identical along the remaining d_1 dimensions. In the continuum limit, i.e., ϵ expansion, the value of d_1 will completely define the correlations between various random bonds, whereas on Bravais lattices (discrete) specifying d_1 is not sufficient, since there are several ways to assign the random bonds. This will be described in what follows. In order to construct lattices with such quenched disorder, four possibilities are considered.

(i) We start by choosing one of the lattice directions, for example, the $\hat{z}$ axis in Fig. 1(a). A constant value is assigned to all bonds pointing in the other $d-1$ directions [e.g., the $\hat{x}$ and $\hat{y}$ directions in Fig. 1(a)]. We will choose all the bonds along the $\hat{z}$ axis to be equal to one another if they belong to the same line and the quenched disorder is introduced by having bonds which vary randomly from line to line according to the following rule: each line is repeated along d_1-1 prechosen directions, so that there are no correlations among lines along the other $d-d_1$ directions. In Fig. 1(a), the lines that are repeated along the $\hat{x}$ axis are identical and those that are repeated along the $\hat{y}$ axis are independent, namely $d_1=2$. Notice that in this case the prechosen $\hat{z}$ axis is always one of the d_1 directions because the lines are composed of identical bonds. An example of this type of correlated randomness in $d=2$ is a model that is the dual of a McCoy-Wu random model[16] where there are constant horizontal lines of bonds and the bond value along the vertical lines changes randomly from line to line but is the same within the same line. Clearly, here $d_1=1$. For the construction described above, we expect two correlation lengths, $\xi_{\|}$ and $\xi_{\perp}$, and the free energy will scale with the exponent $2-\alpha=d_1\nu_{\|}+(d-d_1)\nu_{\perp}$.

(ii) In this case the procedure that was used in (i) is repeated for lines along all the d directions, and the d_1 directions are chosen with cyclic permutations. This is shown in Fig. 2(a) with $d=2$, $d_1=1$. Since the d axes are now treated symmetrically, we expect the correlation length to be the same in all directions, and $2-\alpha=d\nu$.

(iii) This is similar to (i), with the difference that each line itself is composed of independent random bonds and the correlations are in the d_1 of the other $d-1$ orthogonal directions. This is illustrated in Fig. 3(a) where $d=3$ and $d_1=1$ were chosen. In $d=2$, this reduces to the correlated random Ising model that was solved exactly by McCoy and Wu.[16] In their model the lattice is composed

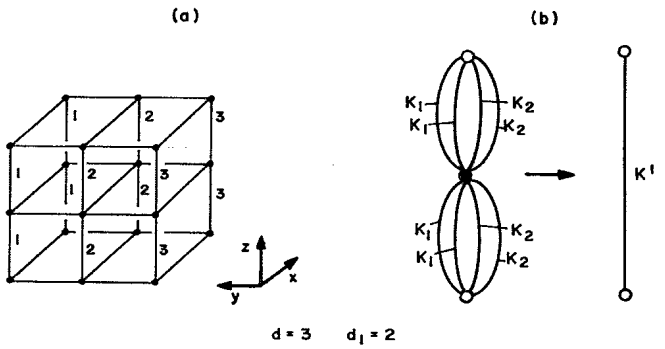

FIG. 1. (a) An example of the case (i) of Sec. II with $d=3$, $d_1=2$. The bonds in the xy plane have a constant value. The bonds in the $\hat{z}$ direction have the same value if they belong to the same xz plane. This value varies randomly from one xz plane to another and is represented by the numbers 1, 2, 3, etc. (b) The hierarchical lattice on which the Migdal-Kadanoff scheme is exact. It corresponds to the RG of the $\hat{z}$-direction bonds of (a).

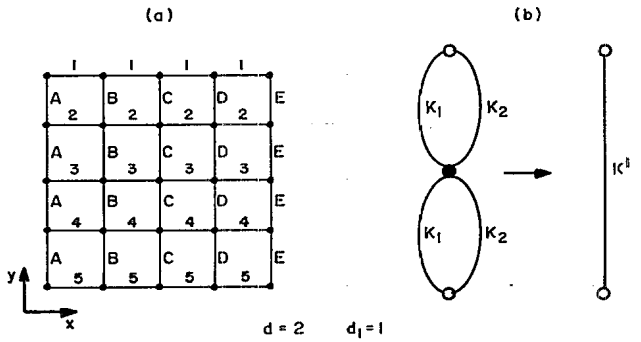

FIG. 2. (a) An example of case (ii) of Sec. II with $d=2$, $d_1=1$. Notice that the lines along the $\hat{y}$ direction are repeated along the $\hat{x}$ direction and also vice versa. For general (d,d_1) this is done with cyclic permutations. The different bonds along the $\hat{x}$ direction are labeled: 1,2,3,4,5, . . . and those along the $\hat{y}$ direction: A, B, C, D, E, ... (b) The hierarchical lattice that corresponds to (a), for which the Migdal-Kadanoff scheme is exact.

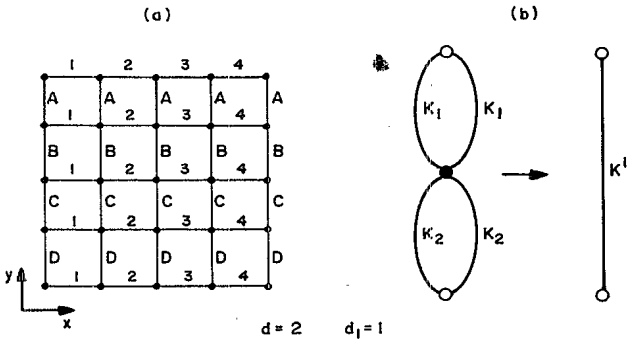

FIG. 4. (a) An example of case (iv) of Sec. II with $d=2$, $d_1=1$. The bonds along each line in the $\hat{y}$ direction are randomly distributed, but the entire line is repeated along the $\hat{x}$ direction. The same procedure is also used for the $\hat{x}$-direction bonds (and in general with cyclic permutations). (b) The hierarchical model that corresponds to (a), for which the Migdal-Kadanoff is exact.

of one random vertical line which repeats itself along the $\hat{x}$ axis and of constant horizontal bonds.

(iv) We can yet construct another model by cyclicly permuting the d_1 directions of (iii), in the same way that was done in (ii), and this is shown in Fig. 4(a) for a lattice with $d=2$, $d_1=1$.

The above discussion applies to any nearest-neighbor model Hamiltonian. For the purposes of the present paper we present results on the q-state Potts model,[20]

$$\overline{\mathcal{H}}=-\mathcal{H}/k_BT=\sum_{\langle i,j\rangle}K_{ij}\delta_{s_is_j}\,, \tag{2.1}$$

where $s_i=1,2,\ldots,q$ and the random K_{ij}'s have correlations as described above.

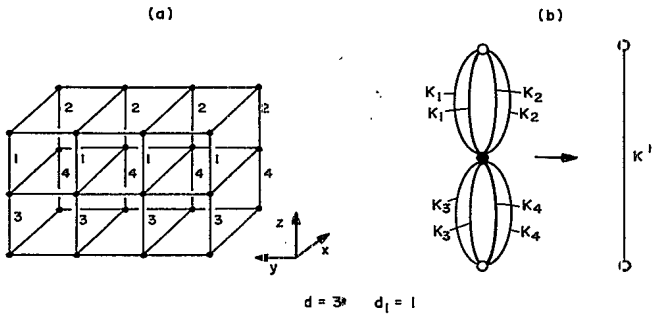

FIG. 3. (a) An example of case (iii) of Sec. II with $d=3$, $d_1=1$. The bonds in the xy plane have a constant value. The bonds along the $\hat{z}$ direction are repeated along the $\hat{y}$ direction but vary randomly in the xz plane. (b) The hierarchical lattice on which the Migdal-Kadanoff scheme is exact. It corresponds to the RG of the $\hat{z}$-direction bonds of (a).

III. THE CROSSOVER EXPONENT

Using the ϵ expansion, Lubensky[3] showed that the crossover exponent from the pure to this correlated random behavior is given by Eq. (1.2). The same results can be obtained using general scaling arguments as outlined in Ref. 7. This is a modification of the usual Harris criterion[2] for the isotropic disorder, where the crossover exponent is just α.[2-9] Therefore there are systems where the isotropic disorder is irrelevant, $\alpha<0$, but nevertheless the correlated disorder is relevant when $\phi=\alpha+d_1\nu>0$, ν being positive for second-order phase transitions. A simple derivation of the Harris criterion for systems with isotropic bond disorder was given by Kinzel and Domany,[8] using a position-space RG method. We will generalize this derivation to include correlated disorder after reviewing it for the isotropic case. Starting with the assumption that each renormalized bond K' depends only on $n=b^d$ original bonds $\{K_i\}$, $i=1,\ldots,n$, where b and n are the length and volume rescaling factors, respectively, the recursion relation for K is

$$K'=R(K_1,K_2,\ldots,K_n)\,. \tag{3.1}$$

The pure critical point is obtained by setting all $K_i=K$ and finding the fixed point $K'=K=K^*$ of Eq. (3.1). The pure thermal exponent $y_T=1/\nu$ is given by

$$b^{y_T}=\sum_{i=1}^{n}\frac{\partial R}{\partial K_i}\bigg|_{K_j=K^*}\,. \tag{3.2}$$

Near the pure critical point, the $\{K_i\}$ are taken to be random independent variables which are characterized by a mean $\langle K\rangle_{\rm av}\simeq K^*$ and a small variance Δ_K, such that

$$\langle K_iK_j\rangle_{\rm av}-\langle K\rangle_{\rm av}^2=\Delta_K\delta_{ij}\,. \tag{3.3}$$

Expanding Eq. (3.1) up to first order in $\delta K_i=K_i-K^*$ leads to

$$\delta K' \equiv K' - K^* = \sum_{i=1}^{n} \left. \frac{\partial R}{\partial K_i} \right|_{K^*} \delta K_i \ . \tag{3.4}$$

Therefore the variance of the renormalized distribution is

$$\Delta_{K'} \equiv \langle (\delta K')^2 \rangle_{\rm av} = \sum_{i,j=1}^{n} \left. \frac{\partial R}{\partial K_i} \right|_{K^*} \left. \frac{\partial R}{\partial K_j} \right|_{K^*} \langle \delta K_i \delta K_j \rangle_{\rm av}$$
$$= \sum_{i=1}^{n} \left[\left. \frac{\partial R}{\partial K_i} \right|_{K^*} \right]^2 \Delta_K \ . \tag{3.5}$$

Note that off-diagonal correlations, $\langle \delta K_i' \delta K_j' \rangle_{\rm av}$, with $i \neq j$, are generated if K_i' and K_j' depend on the same original bond K_i. This does not happen for our Migdal-Kadanoff recursion relations, as discussed below. If the recursion relation, Eq. (3.1) is invariant under permutation of its arguments,[8,9] then Eq. (3.2) yields $\partial R/\partial K_i \mid_{k^*} = b^{y_T - d}$, and

$$\Delta_{K'} = b^{2y_T - d} \Delta_K \ . \tag{3.6}$$

Thus, the crossover exponent for the isotropic disorder is $\phi = (2y_T - d)/y_T$. Using the hyperscaling relation $2 - \alpha = d/y_T$ yields $\phi = \alpha$ as was originally found by Harris.[2]

Returning to the correlated disorder, for every $n = b^d$ bonds there are only $m \equiv n/n_1 = b^{d-d_1}$ independent bond variables, where $n_1 \equiv b^{d_1}$ is the number of repetitions for each independent bond. From Eq. (3.5), the renormalized variance can be expressed as

$$\Delta_{K'} = \sum_{i=1}^{n} \left. \frac{\partial R}{\partial K_i} \right|_{K^*} \sum_{j=1}^{n} \left. \frac{\partial R}{\partial K_j} \right|_{K^*} \langle \delta K_i \delta K_j \rangle_{\rm av}$$
$$= \sum_{i=1}^{n} \left[\left. \frac{\partial R}{\partial K_i} \right|_{K^*} \right]^2 b^{d_1} \Delta_K = b^{2y_T - d + d_1} \Delta_K \ . \tag{3.7}$$

Thus the crossover exponent for this case is

$$\phi = (2y_T - d + d_1)/y_T = \alpha + d_1 \nu \ , \tag{3.8}$$

confirming Eq. (1.2).

IV. RECURSION RELATIONS AND FIXED DISTRIBUTIONS

When $\phi > 0$, the randomness is relevant and the critical behavior is characterized by a fixed distribution $P^*(\{K\})$, where $\{K\}$ is the set of all the bonds. We consider only distributions where the random bonds are independent random variables, i.e., $P(\{K\}) = \prod p(K_i)$. As noted above, no correlations among the K_i's are generated within the Migdal-Kadanoff RG, so that we also have $P'(\{K'\}) = \prod p'(K_i')$. For real Bravais lattices, such a factorization is an approximation, similar to the truncation of longer-range interactions that are generated in the RG of the pure system. Notice that for the particular correlated randomness, described in Sec. II, we should always choose a RG procedure that preserves the structure of the correlated randomness. The neglected additional correlations are only between the *independent bonds*. To make this more clear, we label the independent bonds inside a rescaled volume by $\{K_\alpha,\ 1 \leq \alpha \leq m = b^{d-d_1}\}$ whereas $\{K_i,\ 1 \leq i \leq n = b^d\}$ are *all the bonds* in the same rescaled volume. The recursion relation $K' = R(K_1, \ldots, K_i, \ldots, K_n)$ can also be written as $K' = \tilde{R}(K_1, \ldots, K_\alpha, \ldots, K_m)$ where $\tilde{R}$ is the recursion relation that depends only on the m independent bonds. Now we are able to write the renormalized bond distributions $p'(K')$ as an integral over the original distributions $p(K_\alpha)$, $\alpha = 1, \ldots, m$,

$$p'(K') = \int \left[\prod_{\alpha=1}^{m} dK_\alpha p(K_\alpha) \right] \times \delta(K' - \tilde{R}(K_1, \ldots, K_\alpha, \ldots, K_m)) \ . \tag{4.1}$$

Even in its simplified form, it is extremely difficult to find the exact solution for Eq. (4.1) and we will rely on a numerical approach that was introduced in an earlier paper.[9] This method uses a multidimensional version of the Newton-Raphson algorithm in order to determine the fixed distribution $p^*(K)$ for successively finer numerical representations. Within a given representation, the fixed distribution is approximated by an histogram with M intervals:

$$p_M(K) = \sum_{i=1}^{M} p_i \Delta_i(K) \ , \tag{4.2}$$

where $\Delta_i(K)$ equals 1 if K falls into the ith interval, and is zero otherwise. The range of the bond strengths is divided into M intervals and if the range is $[0, \infty)$ then an upper cutoff K_u is used so that the last interval acts as an integrated tail of the distribution. Each interval is centered at a_i and has a width b_i. The functional recursion (4.1) can now be written as an algebraic recursion relation for the histogram probabilities $\{p_i\}$,

$$p_i' = b_i^{-1} \sum_{l_1=1}^{M} \cdots \sum_{l_m=1}^{M} \Delta_i(\tilde{R}(K_{l_1}, \ldots, K_{l_m})) \times \prod_{j=1}^{m} b_{l_j} p_{l_j}, \quad i = 1, \ldots, M \ . \tag{4.3}$$

The Newton-Raphson algorithm is then applied to solve the $M-1$ coupled nonlinear equations, Eq. (4.3), for the fixed-point condition $p_i' = p_i = p_i^*$. (The normalization condition $\sum_{i=1}^{M} p_i b_i = 1$ is conserved by the RG and can be regarded as an additional constraint for the M variables p_i, thus only $M-1$ of the p_i's will be linearly independent.) Eigenvalue analysis is then performed on the linearized version of Eq. (4.3). A more detailed discussion on this procedure can be found in Ref. 9.

V. THE MIGDAL-KADANOFF RG METHOD

The Migdal-Kadanoff (MK) approximation[22,23] is by far the simplest position space RG method available today. Even though it is only an approximation for Bravais lattices, it can be viewed as an exact decimation procedure on hierarchical lattices.[24] We used the MK method because of its simplicity and the fact that it gives the correct

crossover exponent ϕ for isotropic and for correlated disorders. It will be shown in this section that the MK satisfies all the general requirements of Sec. III and thus giving the correct crossover exponent.

Two steps are involved in carrying out the MK procedure: the first step is to move some of the original bonds from their positions, and to add them to bonds in other locations of the lattice. In the correlated disorder, case (iv) of Sec. II, this means that each one of the remaining bonds is

$$\tilde{K}_\alpha = \sum_{i_\alpha=1}^{m/b} b^{d_1} K_{i_\alpha}, \quad \alpha = 1, \ldots, b \ . \tag{5.1}$$

Note that for case (iii) of Sec. II, Eq. (5.1) holds only for bonds along one of the d_1 directions. In the other $d-1$ directions $\tilde{K} = b^{d-1}K$, since K is a nonrandom bond. The second step, which is exact, is to decimate over the intermediate spins that have one-dimensional connectivity, thus a renormalized lattice is obtained with bond strength K':

$$K' = \tilde{R}(K_1, \ldots, K_m) = F(\tilde{K}_1, \ldots, \tilde{K}_b) \ , \tag{5.2}$$

where $F(\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_b)$ is the resulting bond strength that is obtained after we carry out an exact decimation of the intermediate spins $1,2,\ldots,b-1$ of a one-dimensional chain with b bonds $\tilde{K}_1, \tilde{K}_2, \ldots, \tilde{K}_b$. For $b=2$, $d=3$, and $d_1=1$ we can see from Fig. 3(b) that

$$K' = \tilde{R}(K_1, \ldots, K_4) = F(2(K_1+K_2), 2(K_3+K_4)) \ . \tag{5.3}$$

The renormalized lattice has the same structure and most importantly the same type of correlations between the renormalized bonds as did the original lattice.

We can also use the MK method for cases (i) and (ii) of Sec. II, and the resulting renormalized bonds are

$$\tilde{K} \equiv \tilde{K}_\beta = \sum_{\alpha=1}^{m} b^{d_1-1} K_\alpha, \quad \beta = 1, \ldots, b \tag{5.4}$$

and

$$K' = \tilde{R}(K_1, \ldots, K_m) = F(\tilde{K}_1, \ldots, \tilde{K}_b) = F(\tilde{K}, \ldots, \tilde{K}) \ , \tag{5.5}$$

for all the bonds in case (ii) and only for bonds along one of the d_1 directions in case (i). For example, when $b=2$, $d=3$, and $d_1=1$ then

$$K' = F(K_1+K_2+K_3+K_4, K_1+K_2+K_3+K_4) \ , \tag{5.6}$$

which of course differs from Eq. (5.3). Also we can see from Fig. 1(b) that when $b=2$, $d=3$, and $d_1=2$ then Eqs. (5.4) and (5.5) reduce to

$$K' = F(2(K_1+K_2), 2(K_1+K_2)) \ . \tag{5.7}$$

As can easily be verified, both recursions Eqs. (5.2) and (5.5) satisfy all the assumptions of Sec. III, i.e., each renormalized bond depends on n original bonds of which m are independent. Thus the general derivation of the crossover exponent given in Sec. III holds for the MK approximation. Another important observation for the MK method is that bonds that point in different directions do not mix in the recursion relation, i.e., the renormalized bonds that are along the $\hat{x}$ direction will only depend on original bonds that are also along this direction and the same is true for all other directions. Thus depending on the type of randomness, we will or will not get two correlation lengths in the system. Cases (i) and (iii) are examples where there are two different exponents. Even within the MK approximation we will have two different thermal exponents: one will be the correlated-random exponent and the other will be the regular pure thermal exponent. In cases (ii) and (iv) we will have, because of the permutational symmetry, only one correlation-length exponent, since the recursions of the random bonds in all directions are identical. This is also the behavior that we expect for correlated disorder on Bravais lattices, and in this sense the MK approximation mimics the true behavior. Nevertheless one of the weaknesses of the MK approximation is that because of an effective dimensional decoupling in the scaling form for the free energy, we could not generalize the hyperscaling relation for the specific-heat exponent, $\alpha = 2-d_1\nu_{||}-(d-d_1)\nu_\perp$, even though we do obtain two different thermal exponents whenever we should.

VI. RESULTS FOR $d_1 < d-1$

As indicated in the introduction, we found different types of results for $d_1 = d-1$ and $d_1 < d-1$. We start by describing a particular case of the latter, i.e., $d=3$ and $d_1=1$ (Fig. 3). We substituted Eqs. (5.1)—(5.3), corresponding to cases (iii) and (iv) of Sec. II, into Eq. (4.3), and solved for the fixed distribution for various values of q. The method can equally be applied to the disorder characterized by Eqs. (5.4)—(5.6).

For all q values where $\phi = \alpha + 2\nu > 0$ (i.e., $q > 3$ within our approximate RG), we found a new fixed distribution $p^*(K)$ with a finite width that characterized the random critical behavior (see Table I). The fixed distribution is obtained numerically, with $M=16$, as was described in Sec. IV, and in Fig. 5 we plot it for $q=3$ (the last data point represents the tail for $K > 0.8$). The eigenvalues, found by linearizing the recursion relation (4.3) around the new fixed distribution, are tabulated in Table I. Only the leading eigenvalue is relevant, $y_1 > 0$ and all the others are irrelevant, $y_i < 0$, $i > 1$. Both the mean and the variance of $p^*(K)$ grow as q increases. It can also be seen that as α and ϕ of the pure system decrease, y_1 increases. Basically the results for $d_1 < d-1$ are similar to those for the isotropic disorder, $d_1 = 0$, but the numerical values are quite different, and the correlated disorder will be relevant for small q values where the isotropic disorder is still not relevant. For $q=2$ we have $\phi \simeq -0.128$, and the pure δ-function distribution is recovered.

Note that we can interpret our results for the fixed distribution for both cases (iii) and (iv). In case (iii), MK gives two different correlation-length exponents: one is $\nu_{||} = y_1^{-1}$ from the fixed distribution of the bonds along the $\hat{z}$ axis and the other is $\nu_\perp = y_T^{-1}$, which is exactly like

TABLE I. Random and pure criticality of the q-state Potts models under the $b=2$ Migdal-Kadanoff recursion for $d=3$, $d_1=1$, Eq. (5.3). The fixed distribution with mean $\langle K\rangle^*_{av}$ and standard deviation σ^* were obtained with $M=16$ histograms of equal width using an upper cutoff $K_u=0.8$. y_1 and y_2 are the two leading eigenvalues at random criticality. The exponents $\alpha_1=2-d_1/y_1+(d-d_1)/y_T$ and $\alpha_2=2-d/y_1$ are the specific-heat exponents for the two models discussed in Sec. VI. $1/K_p^*$, y_T, α, and ϕ are the critical temperature, the thermal exponent, the specific-heat exponent, and the crossover exponent for the pure model, respectively. Note that for $q=2$ we get a good agreement with pure criticality even though with our discretization, the fixed distribution has a finite width.

	Random criticality							Pure criticality			
q	$\langle K\rangle^*_{av}$	σ^*	$\sigma^*/\langle K\rangle^*_{av}$	y_1	y_2	α_1	α_2	K_p^*	y_T	α	ϕ
2	0.136	0.080	0.588	0.939	−0.037	−1.193	−1.193	0.131	0.939	−1.193	−0.128
3	0.170	0.147	0.865	0.916	−0.148	−1.043	−1.275	0.165	1.025	−0.927	0.049
4	0.201	0.196	0.975	0.890	−0.364	−0.957	−1.370	0.192	1.091	−0.750	0.167
5	0.225	0.218	0.969	0.885	−0.399	−0.878	−1.391	0.215	1.144	−0.623	0.251
6	0.243	0.234	0.963	0.877	−0.427	−0.824	−1.422	0.234	1.188	−0.526	0.316
7	0.262	0.248	0.947	0.873	−0.517	−0.777	−1.441	0.251	1.226	−0.447	0.369

the exponent of the pure system. In Table I, we give the two corresponding specific heat exponents: $\alpha_1=2-d_1/y_1-(d-d_1)/y_T$ for case (iii) and $\alpha_2=2-d/y_1$ for case (iv).

VII. RESULTS FOR $d_1=d-1$

When $d_1=d-1$, we have correlations between the random bonds in all but one dimension (an example is the McCoy-Wu model on the square lattice). Consider the randomness that is discussed in case (iii) of Sec. II with $d_1=d-1$. If the bonds have a finite probability to have a zero value, namely

$$p(K)=p_0\delta(K)+\tilde{p}(K)\,, \tag{7.1}$$

then the system will be cut into an infinite number of disconnected pieces, each piece being infinite only along the single direction along which there are no correlations. This means that there is no phase transition for this system at finite temperatures. This fact can be seen also from the MK recursion, since from Eqs. (5.1) and (5.2) $K'=F(b^{d-1}K_1,\ldots,b^{d-1}K_b)=0$ if any of the K_i's is zero. Therefore all the distributions of the form (7.1) will renormalize to the trivial sink, P in Fig. 6(a), for which $p^*(K)=\delta(K)$, since the renormalized weight of the δ function at zero will always grow. The projected flow diagram for this case is shown schematically in Fig. 6(a), in which σ denotes the width of the distribution, i.e.,

$$\sigma^2=\Delta_K=\langle K^2\rangle_{av}-\langle K\rangle^2_{av}\,. \tag{7.2}$$

Since our discretization, Eq. (4.2) gives a finite weight to a region around $K=0$, it is clear that $p_0\neq 0$ will always be generated, and thus no fixed distribution with finite $\langle K\rangle^*_{av}$ can be found. The flows would rather follow Fig. 6(a). In order to avoid this problem, we introduced a lower cutoff, K_l. The fixed distributions that were then

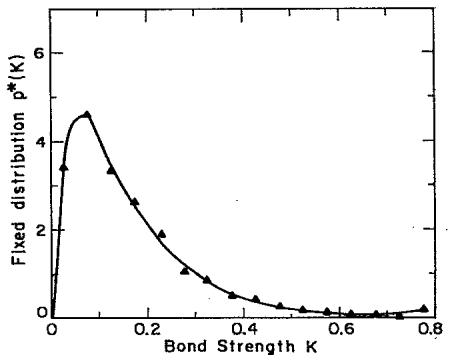

FIG. 5. Fixed distribution of the $q=3$ (▲) Potts model under the Migdal-Kadanoff recursion with $b=2$, $d=3$, $d_1=1$, Eqs. (5.1) and (5.2). The bond strength range $0\leq K\leq 0.8$ is divided into $M=16$ intervals of equal width. The tail $K>K_u=0.8$ is integrated and added to the last interval. The curve between the data points has only presentational purpose.

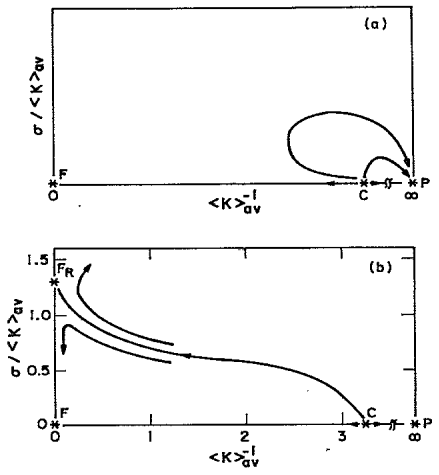

FIG. 6. Projections of the flows for $d=2$, $d_1=1$, Eqs. (5.1) and (5.2), into a two-dimensional space of the first two moments of the distribution: $\langle K\rangle^{-1}_{av}$ and $\sigma/\langle K\rangle_{av}$. In (a) our starting distribution has a finite weight at $K=0$: $p(K)=p_0(K)\delta(K)+\tilde{p}(K)$. The only fixed points are the three pure fixed points $F(K=\infty)$, $C(K=K_c)$, and $P(K=0)$. The flows are drawn schematically. In (b) we started with a Γ distribution $p(K)\sim K^{\alpha-1}\exp(-\beta^{-1}K)$. The calculated critical line starts from the pure critical point C and approaches a new fixed point F_R at $K=\infty$ which is different from the pure fixed point F at $K=\infty$.

found were very sensitive to the value of K_l; as $K_l \to 0$ the average $\langle K \rangle_{\rm av}^*$ and all the moments (e.g., σ) approached infinity, with what seemed to be finite ratios among them. The corresponding limiting fixed distribution is denoted by F_R in Fig. 6(b), which we believe describes the flow in the general cases $d_1 = d-1$ (we checked it explicitly for $d=2$, 3).

Because of the difficulty with the lower cutoffs, we also studied the flows by repeatedly integrating (numerically) Eq. (4.1) for $d=2$, $q=2$ (Ising), without the use of our discretization procedure. Starting with a Γ distribution,

$$p(K) \sim K^{\alpha-1} \exp(-\beta^{-1} K) , \tag{7.3}$$

with $\langle K \rangle_{\rm av} = 5.0$, we found a critical value $\sigma = 4.9331$ for which the flow indeed followed Fig. 6(b). (The entire C-F_R line is obtained from other initial conditions as well.) As the point F_R was approached, we indeed found that $\langle K \rangle_{\rm av}$, $\sigma \to \infty$ but $\sigma / \langle K \rangle_{\rm av}$ remained finite. Although F_R occurs at $T=0$, it is different from the ferromagnetic sink fixed point F, where only $\langle K \rangle_{\rm av} = \infty$, while σ and higher moments at zero.

Our numerical integration was not accurate enough to determine the thermal exponent y_T close to F_R. However, crude estimates indicate that $\alpha < 0$, as expected from $\alpha = -\infty$ of McCoy and Wu, and in agreement with Ref. 10. Estimates of the specific-heat exponent were also obtained by a discretization on the $\tanh K$ variable, with a lower cutoff K_l that ranges between 0.025 to 0.050. We found that $\alpha_1 = 2 - 1/y_1 - 1/y_T \simeq -1.8$, and $\alpha_2 = 2 - 2/y_1 \simeq -3.0$.

We found it easier to determine the magnetic exponent, y_h, near F_R which can be calculated by averaging the following expression over the critical distribution (near F_R):

$$b^{y_h} = \left\langle \frac{\partial h'}{\partial h} \right\rangle_{\rm av} . \tag{7.4}$$

The value for y_h found when we approach F_R is $y_h = 1.996$ for the $d=2$, $d_1=1$ Ising model. Since it is so close to $y_h = d = 2$ it shows that F_R is a discontinuity fixed point, and that along the whole line C-F_R (except the point C) we have a first-order phase transition with a discontinuous onset of magnetization ($y_h = d$), but no latent heat ($y_1 \simeq 0.4$). We are not aware of any other calculation that predicts this behavior for correlated disorder. It is interesting to note that, on the one hand, first-order transitions are usually accompanied by essential singularities.[28] On the other hand, McCoy and Wu[16] found such an essential singularity in the square Ising model with axial randomness. This raises, by analogy, the novel possibility of a discontinuous onset of magnetization at the McCoy-Wu transition.

Some support for our flow structure is found in a recent paper by Collet *et al.*[29] They studied the flow of the distribution function of an Ising spin glass on a hierarchical lattice, with a discretization in which the interval width varies as $b_l \sim 2^{-l}$ [Eq. (4.3)]. Their results indicate a surface in the distributions space above which the flows go to $T = \infty$. The points on and below this surface flow to $T=0$, as also found by us.

VIII. CONCLUSIONS

Unlike the perturbative calculations of the correlated random Potts models,[21] we have been able to find fixed random distributions for $d_1 < d-1$ whenever $\phi = \alpha + d_1 \nu > 0$. These new distributions have well-defined critical exponents, and their correlation lengths depend on the symmetry of the random correlations. It remains to be seen if the accounting of effective vacancies may change these continuous transitions into first-order ones.[25]

We find more unusual results for $d_1 = d-1$, when the random behavior is controlled by a novel zero-temperature fixed distribution, with a magnetic exponent $y_h = d$ which indicates a first-order transition and associated essential singularities.

It will be very interesting to understand the connection between this phase diagram which is exact within the MK scheme, and the essential singularity in the McCoy-Wu model, and whether this essential singularity is characteristic to all models with $\phi > 0$ and $d_1 = d-1$. Notice that when $\phi < 0$, the flows on the critical surface are reversed and the pure critical point $K_c = K^*$ will determine the critical behavior.

Also of interest would be to check the qualitative and quantitative features of the correlated random models in real experiments. A candidate system is the random quantum Ising magnet $LiTb_pY_{1-p}F_4$, which exhibits a zero-temperature transition for $p \simeq 0.21$.[30]

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