J. Phys. A: Math. Gen. 13 (1980) L413–L418. Printed in Great Britain

LETTER TO THE EDITOR

First- and second-order phase transitions of infinite-state Potts models in one dimension

A N Berker, D Andelman and A Aharony†

Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

Received 27 August 1980

Abstract. The q-state, d-dimensional Potts models exhibit a variety of phase-transition behaviour in the limit $d \to 1^+$, $q \to \infty$, and $l \equiv (d-1) \ln q$ finite. The regions $l < 1$, $1 < l < 2$, and $2 < l$ are distinguished, respectively, by no transition, second-order transitions (with a new changeover phenomenon at $l = \ln 4$), and first-order transitions. The latter are due to the condensation of effective vacancies. Critical and tricritical exponent values are given.

Experimental realisations (Alexander 1975, Aharony *et al* 1977) and connections with percolation (Fortuin and Kasteleyn 1969) are reasons for the current high interest in the phase transitions of the q-state Potts (1952) models. Another reason is a sustained conceptual development for spatial dimension $d = 2$, from rigorous (Baxter 1973, 1979), conjectural (den Nijs 1979, Nienhuis *et al* 1979, 1980b, Nauenberg and Scalapino 1980) and approximate (Nienhuis *et al* 1979, 1980a, b) theories. These theories show that the phase transitions of the Potts models change from second to first order at $q = q_c(d)$, with $q_c(2) = 4$ (Baxter 1973). For $q < q_c$, the second-order transitions can also turn first order at a tricritical point, for a sufficiently high concentration of vacancies (Berker *et al* 1978). The critical and tricritical behaviours coalesce at q_c, and first-order transitions for $q > q_c$ result from a condensation of effective vacancies (Nienhuis *et al* 1979).

The relevance to experimental systems in $d = 3$ (Aharony *et al* 1977) makes it highly desirable to understand the d-dependence of q_c. It has recently been shown exactly (Aharony and Pytte 1980) that $q_c(d) = 2 + \epsilon + O(\epsilon^2)$ at $d = 4 - \epsilon$, but we are not aware of any other exact result at $d \neq 2$. One should hope that a combination of exact results at various values of d should make possible a reliable interpolation for q_c of $d = 3$. An approximate calculational scheme in which d can be continuously varied is also highly desirable. The present work reports a particular form of the Migdal–Kadanoff (Migdal 1975, Kadanoff 1976) renormalisation method which successfully fulfils this goal. In particular, our calculations seem to be either exact, or accurate in the limit $d \to 1^+$, $q \to \infty$ with $l \equiv (d-1) \ln q$ fixed and finite. In this limit, a new variety of (zero-temperature) phase-transition behaviour is found and conveniently studied.

The model exhibits no phase transition for $l < l_0 = 1$, and first-order transitions for $l > l_c = 2$ due to condensation of effective vacancies. In the range $l_0 < l < l_c$, we find critical and tricritical behaviours which vary smoothly with l. The critical behaviour has a non-analyticity at $l_1 = \ln 4$, where the 'undiluted' fixed point becomes unstable as a 'diluted' one emerges from it. The critical and tricritical behaviours coalesce at l_c. The

† Permanent address: Tel Aviv University, Israel.

0305-4470/80/110413+06\$01.50

special values $l_0 = 1$ and $l_c = 2$ are independent of length-rescale factor b, and therefore appear to be exact. This implies that $q_c(d)$ increases as $\exp[2/(d-1)]$ for $d \to 1^+$. Similarly, the tricritical exponents given below appear to be exact. The critical exponents, as well as the special value $l_1 = \ln 4$, appear to be accurate (that is, insensitive to changes in b). The results for the various exponents and fixed points are presented in figure 1. The renormalisation-group recursion relations given below are also useful for all $1 < d \leqslant 4$ (Andelman, Berker and Aharony 1980, in preparation). They reproduce types of phase diagrams found before only for $d = 2$ (Nienhuis *et al* 1979). As an example, in figure 2 are results for $q_c(d) = 4$.

Our new results should help in studying the behaviour of the Potts models in $d > 1$, where analogues of the special points l_0 and l_1 are found (Andelman, Berker and Aharony 1980, in preparation). We also hope to stimulate discussion of the new types of behaviour in the one-dimensional limit and of the possibility of their exact nature.

The renormalisation-group study (Nienhuis *et al* 1979) of Potts models is within the context of the Potts lattice-gas (Berker *et al* 1978),

$$\frac{-\mathscr{H}}{kT} = J \sum_{\langle ij \rangle} t_i t_j (\delta_{s_i s_j} - 1) - \frac{F}{2} \sum_{\langle ij \rangle} (t_i - t_j)^2 + G \sum_i t_i. \tag{1}$$

At each site i of a hypercubic lattice, the Potts variable $s_i = a, b, c, \ldots$ can take one of q values, $\delta_{s_i s_j} = 1(0)$ for $s_i = s_j (s_i \neq s_j)$, and the lattice-gas variable $t_i = 1(0)$ corresponds to an occupied (vacant) site. The sums $\langle ij \rangle$ are over all pairs of nearest-neighbour sites. In the limit $G \to \infty$, all vacancies are removed from the system, and a conventional, undiluted Potts model is recovered. However, it is important that, in many regimes, this undiluted system is thermodynamically equivalent to a diluted Potts model with an increased length scale (Nienhuis *et al* 1979).

Our central results are best discussed with figures $1(a, b)$. For $l < 1$, there is no ordered phase. For $l > 2$, the phase transition between the ordered ($\langle \delta_{s_i a} \rangle \neq 0$) and disordered phases is of first order. For any l between 1 and 2, a surface of critical points constitutes the phase boundary in the space of interactions (J, F, G). This surface is bounded by a line of tricritical points. The critical points include the transition of the undiluted Potts model. The critical (y_i^c) and tricritical (y_i^t) exponents are given in figures $1(a, b)$. The exponent y_2 is the inverse of the correlation length exponent, $y_2 = \nu^{-1}$, and gives the leading singularity at the phase transition. The exponent y_4^t gives the exponent of crossover to criticality $\phi = y_4^t / y_2^t$ (Riedel 1972). The negative exponents (irrelevant) y_4^c and $y_6^{c,t}$ give the correction-to-scaling exponents y_i/y_2 (Wegner 1972). Since y_4^c is zero (marginal) at $l_1 = \ln 4$ and $l_c = 2$, logarithmic corrections are expected to power-law singularities (Nauenberg and Scalapino 1980). The discussion of the next paragraph suggests that the tricritical exponents†

$$y_2^t = 1, \qquad y_4^t = 2 - l, \qquad y_6^t = -1, \qquad \text{for } 1 \leqslant l \leqslant 2, \tag{2}$$

are exact, whereas the empirical fits to the critical-exponent† data in figures $1(a, b)$,

$$\left.\begin{aligned} y_2^c &= l - 1 - 0{\cdot}46(l-1)^{1{\cdot}84}, \\ y_4^c &= 3l - 4 - 0{\cdot}92(l-1)^{1{\cdot}84} \qquad \text{for } 1 \leqslant l \leqslant \ln 4, \\ y_2^c &= l - 1 - 7{\cdot}33(2-l)^{9{\cdot}27}, \\ y_4^c &= l - 2 + 2{\cdot}43(2-l)^{2{\cdot}82} \qquad \text{for } \ln 4 \leqslant l \leqslant 2, \end{aligned}\right\} \tag{3}$$

are numerically accurate to the scale of figures 1.

† The scaling fields of $y_4^t (1 \leqslant l \leqslant 2)$ and $y_4^c (1 \leqslant l \leqslant \ln 4)$ are e^{-F} and e^{-G}, respectively.

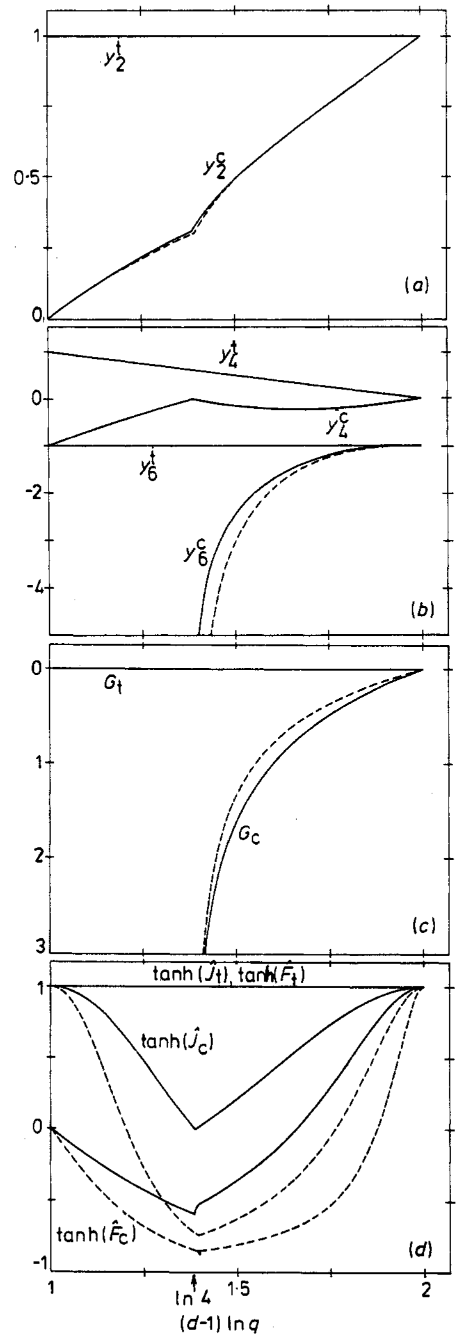

Figure 1. Exponents and fixed-point locations of the Potts models in the limit $d \to 1^+$ and $q \to \infty$. Critical (c) and tricritical (t) properties obtained with length-rescale factor $b = 1{\cdot}01$ are shown with full curves. Results with $b = 4$ are shown with broken curves, when distinguishable. The exponent y_6 is $-\infty$ for $1 \leq (d-1) \ln q \leq \ln 4$.

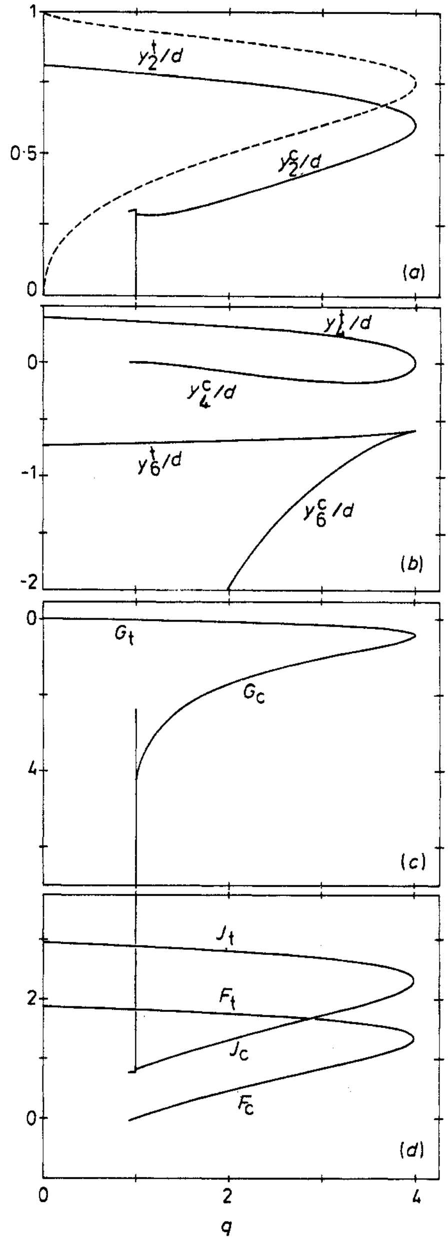

Figure 2. Exponents and fixed-point locations of the Potts models, obtained with the $b = 2$ Migdal–Kadanoff approximation for $d = 1{\cdot}78$, which yields $q_c(d) = 4$. Critical (c) and tricritical (t) properties are shown. The broken curve in (a) is the extended den Nijs (1979) conjecture for $d = 2$, $q_c(d) = 4$. Vertical segments appear at $q = 1$ due to a line of critical fixed points with continuously varying properties.

The Migdal–Kadanoff (Migdal 1975, Kadanoff 1976) renormalisation is effected by first choosing a superlattice composed of hypercubes of side b times the original lattice constant. The couplings (first two terms in equation (1), see comment below) are moved to the edges of these hypercubes, and all degrees of freedom not at hypercube corners are summed over. This procedure is approximate in $d>1$. In the naive limit $d \to 1^+$, it clearly becomes exact. In our combined limit $d \to 1^+$ and $q \to \infty$, its status is less obvious. However, circumstantial evidence here is rather favourable. The renormalisation was carried out for arbitrary length-rescale factor b. The changeover values $l_0 = 1$ and $l_c = 2$, as well as the tricritical properties (equations (2)), turned out to be independent of b. This is characteristic of exact theories (Nelson and Fisher 1975) and suggests the exactness of this subset of information. The changeover value $l_1 = \ln 4$ and equations (3) are obtained in the minimal rescaling limit of $b \to 1^+$. As b is increased, say to 4, ln 4 changes to ln 4·034, and the properties in equations (3) are affected negligibly on the scale of figures 1. This suggests numerical accuracy. On the other hand, y_6^c ($\ln 4 < l < 2$) does change, as seen in figure 1(b), and is therefore less reliable. Odd exponents, evaluated with $b = 2$ only, are $y_1^{c,t} = 1$, $y_3^{c,t} = 0$, $y_5^t = -1$, and y_5^c monotonically decreasing from 0 at l_0 to -1 at l_c, with a change of slope at l_1.

In order to obtain the phase-transition behaviour of Potts models in arbitrary d, we found that it is essential to use the Hamiltonian separation into coupling and on-site terms given in equation (1). This is also the physically most reasonable separation, minimising the effect of the bond-moving approximation both in the ordered and disordered limits (Emery and Swendsen 1977). The resulting recursion relations are

$$\begin{aligned} &x' = 1 + (R_0 - q^{-1})^{-1}, \qquad f' = R_1^{-2} R_2 (R_0 - q^{-1} + 1), \qquad g' = R_2^{-1}(R_0 - q^{-1} + 1),\\ &R_n = \lambda_+^b A_+^n (q + A_+^2)^{-1} + \lambda_-^b A_-^n (q + A_-^2)^{-1},\\ &A_\pm = w \pm (q + w^2)^{1/2}, \qquad w = \tilde{f}^{1/2}[g^{-1/2} - g^{1/2}(1 - \tilde{x}^{-1} + q\tilde{x}^{-1})]/2,\\ &\lambda_\pm = (g^{-1} - \tilde{f}^{-1/2} g^{-1/2} A_\mp)/(1 - \tilde{x}^{-1}), \end{aligned} \tag{4}$$

where ($\ln \tilde{x} = b^{d-1} J$, $\ln \tilde{f} = b^{d-1} F$, $\ln g = G$) and ($\ln x' = J'$, $\ln f' = F'$, $\ln g' = G'$) are respectively the bond-moved and renormalised interactions. The $d \to 1^+$ and $q \to \infty$ limit is taken by introducing $J \equiv \hat{J} + \ln q$ and $F \equiv \hat{F} + \ln q$. The phase diagram structure is in the space ($\hat{J}$, $\hat{F}$, G). The fixed-point locations are given in figures 1(c, d). (A fixed point can be moved within its domain of attraction by changing the transformation, such as by varying b, with no physical consequence.) For $l < l_0$, the ordered sink G^*, $\hat{J}^* = \infty$, $\hat{F}^* = 0$ is unstable, and there is no phase transition. In $l_0 < l < l_c$, the tricritical fixed points, unstable within the phase boundary surface, occur at $G^* = 0$, $\hat{J}^* - \ln 2 = \hat{F}^* = \infty$. In this range, the first-order transitions which are seen beyond the tricritical points for $d > 1$ have shrunk to infinity, as the tricritical points moved to $\hat{F}^* = \infty$ for $d \to 1^+$. In $l_0 < l < l_1$, the critical fixed points, which are stable within the phase boundary, occur at the undiluted limit $G^* = \infty$. In $l_1 < l < l_c$, the critical fixed points traverse the diluted region to annihilate with the tricritical fixed point at l_c. In $l > l_c$, the stable fixed points of the phase boundary are those of first-order phase transitions, at $G^* = 0$, $\hat{J}^* - \ln 2 = \hat{F}^* = \infty$.

The recursion relations (4) are useful approximations in $d > 1$. Figures 2 are the analogues of figures 1, obtained for $d = 1{\cdot}78$ and exhibiting $q_c = 4$. Resemblance between $d \to 1^+$ and $d > 1$ results is noticeable. A full account of our $1 < d \leqslant 4$ results will be given elsewhere (Andelman, Berker and Aharony 1980, in preparation). Nevertheless, we cannot refrain from noting (figures 2) that, at $q = 1$, a line of critical fixed points with continuously varying exponents is found. Thus, the vertical segment in figure 2(a) is the range of these exponents $y_2^c(q=1)$, while $y_4^c(q=1)=0$ is marginal in figure 2(b). This could be of some importance to percolation phenomena (Klein *et al* 1978).

We terminate with a phenomenological discussion of the physical mechanisms behind the changeover at l_c or q_c. Consider low-temperature configurations as shown in figure 3. The unmarked regions represent complete local disorder. For high q, such regions have a multiplicity of about q per site. Clearly, they are entropically favoured at high q. They have no Potts-ordering influence on neighbouring localities, so that they act like clusters of effective vacancies (Nienhuis *et al* 1979) in a grand canonical ensemble, with effective chemical potential $\mu \simeq kT \ln q$. This is reflected by an undiluted Potts model which is unstable under renormalisation ($l > l_1$). As temperature is increased, two mechanisms compete: the surface tension between the ordered domains a and b decreases, and the chemical potential of the vacancies increases. A second- or first-order phase transition occurs, depending on whether the surface tension reaches zero ($l < l_c$), or the vacancies undergo a gas–liquid transition ($l > l_c$), destroying the connectivity of the ordered domains.

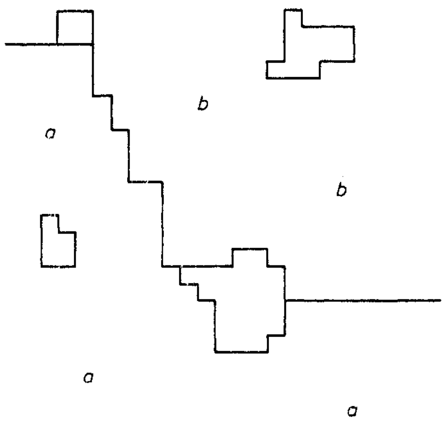

Figure 3. Typical configurations of Potts models which are important at low temperature. Two ordered domains, $s = a$ and b, are shown. The unmarked regions are localities of complete disorder. For high-q models, these are entropically favoured, their condensation causing the first-order phase transition.

This research was supported by the National Science Foundation Grant No DMR-79-26405 and by the United States–Israel Binational Science Foundation.

References

Aharony A, Müller K A and Berlinger W 1977 *Phys. Rev. Lett.* **38** 33
Aharony A and Pytte E 1980 *IBM Preprint*
Alexander S 1975 *Phys. Lett.* A **54** 353
Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445
Baxter R J 1979 *Australian National University Preprint*
Berker A N, Ostlund S and Putnam F A 1978 *Phys. Rev.* B **17** 3650
Emery V J and Swendsen R H 1977 *Phys. Lett.* A **64** 325

Fortuin C M and Kasteleyn P W 1969 *J. Phys. Soc. Japan Suppl.* **26** 11
Kadanoff L P 1976 *Ann. Phys., NY* **100** 359
Klein W, Stanley H E, Reynolds P J and Coniglio A 1978 *Phys. Rev. Lett.* **41** 1145
Migdal A A 1975 *Zh. Eksp. Teor. Fiz.* **68** 1457 (transl. 1976 *Sov. Phys.-JETP* **42** 743)
Nauenberg M and Scalapino D J 1980 *Phys. Rev. Lett.* **44** 837
Nelson D R and Fisher M E 1975 *Ann. Phys., NY* **91** 226
Nienhuis B, Berker A N, Riedel E K and Schick M 1979 *Phys. Rev. Lett.* **43** 737
Nienhuis B, Riedel E K and Schick M 1980a *J. Phys. A: Math. Gen.* **13** L31
—— 1980b *University of Washington Preprint*
den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106
Riedel E K 1972 *Phys. Rev. Lett.* **28** 675
Wegner F J 1972 *Phys. Rev.* B **5** 4529